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Some Properties of the Beurling-Ahlfors Extension*

Sun Zong-liang[1] and Li Shu-long[2]

(*1. Department of Mathematics, Shenzhen University, Shenzhen, Guangdong, 518060*)

(*2. School of Biomedical Engineering, Southern Medical University, Guangzhou, 510515*)

Abstract: In this paper, we study the Beurling-Ahlfors extensions and prove two results. The first variation of the Beurling-Ahlfors extension is not always harmonic; the Beurling-Ahlfors extension of a quasisymmetric mapping is not always harmonic.

Key words: Beurling-Ahlfors extension, first variation, harmonic map

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1 Introduction

McMullen[1] proved that the first variation of the Douady-Earle extension is harmonic. Liu and Yao[2] proved that the Douady-Earle extension is not always harmonic. To find nice extensions of quasiconformal (quasisymmetric) homeomorphisms of $\mathbb{S}^n$ to $\mathbb{H}^n$ is an interesting and important problem. Quasiconformal extensions were first constructed by Beurling and Ahlfors[3] in dimension 2 and higher dimensional extensions were given by Tukia and Välsälä[4]. Tukia[5] constructed a version that was compatible with the action of the group of Möbius transformations. Douady and Earle[6] constructed a conformally natural version in all dimensions, and for any homeomorphism between the circles, its Douady-Earle extension always exists. Hardt and Wolf[7] showed that the set of quasiconformal (quasisymmetric, if $n = 2$) mappings $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ which admit quasiconformal harmonic extensions $F : \mathbb{H}^n \to \mathbb{H}^n$ is open in the set of quasiconformal (quasisymmetric, respectively) self-mappings of $\mathbb{S}^{n-1}$. In [7], the authors also considered the question of finding a harmonic extension which is compatible with Möbius transformations that Royden ever asked. As to this question, Scheon[8] gave a conjecture as follows. For any homeomorphism between the unit circles which admits a quasiconformal extension, there is a harmonic extension on the Poincaré disk (the unit disc with the Poincaré metric). Li and Tam[9,10] first constructed these harmonic extensions under some assumptions of smoothness of the boundary mappings

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$f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ and a lower bound on its energy density (see [11]). Non-uniqueness properties of these extensions were also identified (see [10] and [12]).

It is well known that if the Douady-Earle extension is harmonic, then it coincides with the harmonic extension. Thus it is natural to ask whether the Douady-Earle extension is harmonic. Because if it was, then it would be easier to construct quasiconformal harmonic extensions of quasisymmetric boundary mappings. And, Scheon's conjecture would be true. However, Liu and Yao[3] proved that the Douady-Earle extension is not always harmonic. Besides, McMullen[1] showed that the first variation of Douady-Earle extension is harmonic (Liu and Yao[2] gave a new proof).

In this paper, we consider the following two questions about the Beurling-Ahlfors extension: Is it true that the first variation of the Beurling-Ahlfors extension is always harmonic? Is it true that the Beurling-Ahlfors extension of a quasisymmetric mapping is always harmonic? Our main results are as follows.

Theorem 1.1 *The first variation of the Beurling-Ahlfors extension is not always harmonic.*

Theorem 1.2 *The Beurling-Ahlfors extension of a quasisymmetric mapping is not always harmonic.*

2 Preliminaries

2.1 Harmonic Map and First Variation

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\mathbb{S}^1 = \partial\mathbb{D}$, $\bar{\mathbb{D}} = \mathbb{D} \bigcup \mathbb{S}^1$. Denote by $\Re z$ and $\Im z$ the real part and imaginary part of a complex number z, respectively. Let M be a C^∞ surface. Consider two metrics $\sigma|\mathrm{d}z|^2$ and $\rho|\mathrm{d}\omega|^2$ on M, where z and ω are holomorphic coordinates of M. For an arbitrary Lipschitz mapping $\omega : (M, \sigma|\mathrm{d}z|^2) \to (M, \rho|\mathrm{d}\omega|^2)$, we define the energy density of ω to be

$$e(\omega; \sigma, \rho) = \frac{\rho(\omega(z))}{\sigma(z)}|\omega_z|^2 + \frac{\rho(\omega(z))}{\sigma(z)}|\omega_{\bar{z}}|^2,$$

and the total energy to be

$$E(\omega; \sigma, \rho) = \int_M e(\omega; \sigma, \rho)\sigma \mathrm{d}z\mathrm{d}\bar{z} = \int_M ((\rho(\omega(z))|\omega_z|^2 + \rho(\omega(z))|\omega_{\bar{z}}|^2)\mathrm{d}z\mathrm{d}\bar{z}.$$

It is easy to see that the total energy depends only on the metric $\rho|\mathrm{d}\omega|^2$ on $(M, \rho|\mathrm{d}\omega|^2)$, and in fact, depends only on the conformal structure on $(M, \sigma|\mathrm{d}z|^2)$. We take the above total energy $E(\omega; \sigma, \rho)$ as a functional of ω, and call its critical map $z \mapsto \omega(z)$ a harmonic map. It is well known that the map $z \mapsto \omega(z)$ is harmonic if and only if it satisfies the so called Euler-Lagrange equation

$$T_\omega = \omega_{z\bar{z}} + (\log \rho)_\omega \omega_z \omega_{\bar{z}} = 0.$$

Let $V(\mathbb{S}^1)$, $V(\mathbb{D})$ and $V(\bar{\mathbb{D}})$ be spaces generated by continuous vector fields on $\mathbb{S}^1$, $\mathbb{D}$ and $\bar{\mathbb{D}}$, respectively. For a smooth vector field f in $V(\mathbb{S}^1)$, there exists a family of one parameter