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A Nine-modes Truncation of the Plane Incompressible Navier-Stokes Equations*

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Abstract: In this paper a nine-modes truncation of Navier-Stokes equations for a two-dimensional incompressible fluid on a torus is obtained. The stationary solutions, the existence of attractor and the global stability of the equations are firmly proved. What is more, that the force f acts on the mode k_3 and k_7 respectively produces two systems, which lead to a much richer and varied phenomenon. Numerical simulation is given at last, which shows a stochastic behavior approached through an involved sequence of bifurcations.

Key words: the Navier-Stokes equation, the strange attractor, Lyapunov function, bifurcation, chaos

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1 Introduction

Navier-Stokes equations as very interesting complicated nonlinear equations have been widely studied in the last five decades within many subjects. Such equations always exhibit a rich phenomenology as parameters go through certain values, and it attracts many scientists' attention. In the last century, the scientists like Lorenz and Franceshini did a lot of work on such equations and obtained many valuable achievements (see [1]–[7]). By studying their papers, recently we get a nine-modes model by extending their five-mode equations (see [1]). The new equations exhibit a much richer and varied phenomenology with a large range of Reynolds number. It appears to us how the phenomenology changes with the addition of new modes. Moreover, by putting the force either on k_3 or k_7, the systems both become to be chaos, which enrich the theory of Franceschini and Tebaldi[2].

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2 Nine-modes Lorenz-like Equations

Consider the incompressible Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + f + \nu \triangle u, \tag{2.1}$$

$$\nabla \cdot u = 0, \tag{2.2}$$

$$\int_{T^2} u \mathrm{d}X = 0 \tag{2.3}$$

on the torus $T^2 = [0, 2\pi] \times [0, 2\pi]$, where u is the velocity field, p is the pressure and f is (periodic) volume force. We expand u, p, f in Fourier series on a torus $T^2 = [0, 2\pi] \times [0, 2\pi]$:

$$u(X, t) = \sum_{\boldsymbol{K} \neq \boldsymbol{0}} \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot X} r_{\boldsymbol{K}} \frac{\boldsymbol{K}^{\perp}}{|\boldsymbol{K}|}, \tag{2.4}$$

$$f(X, t) = \sum_{\boldsymbol{K} \neq \boldsymbol{0}} \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot X} f_{\boldsymbol{K}} \frac{\boldsymbol{K}^{\perp}}{|\boldsymbol{K}|}, \tag{2.5}$$

$$p(X, t) = \sum_{\boldsymbol{K} \neq \boldsymbol{0}} \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot X} p_{\boldsymbol{K}} \frac{\boldsymbol{K}^{\perp}}{|\boldsymbol{K}|}, \tag{2.6}$$

where $\boldsymbol{K} = (k_1,\ k_2)$ is a "wave vector", with integer components, $\boldsymbol{K}^{\perp} = (k_2,\ -k_1)$, and the reality condition $r_{\boldsymbol{K}} = -\bar{r}_{-\boldsymbol{K}}$ holds. Substituting (2.4), (2.5), (2.6) into (2.1) we get formally the following equations for $\{r_{\boldsymbol{K}}\}_{\boldsymbol{K}\neq\boldsymbol{0}}$ ($r_{\boldsymbol{K}} = r_{\boldsymbol{K}}(t)$ is a function of t)

$$\dot{r}_{\boldsymbol{K}} = -\mathrm{i} \sum_{\boldsymbol{K}_1+\boldsymbol{K}_2+\boldsymbol{K}=\boldsymbol{0}} \frac{\boldsymbol{K}_1^{\perp} \cdot \boldsymbol{K}_2(\boldsymbol{K}_2^2 - \boldsymbol{K}_1^2)}{2|\boldsymbol{K}||\boldsymbol{K}_1||\boldsymbol{K}_2|} \bar{r}_{\boldsymbol{K}_1} \bar{r}_{\boldsymbol{K}_2} - \nu |\boldsymbol{K}|^2 r_{\boldsymbol{K}} + f_{\boldsymbol{K}} \qquad (\boldsymbol{K} \in L), \tag{2.7}$$

where L is a set of wave vectors, such that if $\boldsymbol{K} \in L$, also $-\boldsymbol{K} \in L$.

In paper [2], a seven modes truncation of (2.7) was got. As the modes increase, it may lead to a much richer and varied phenomenon. But at the same time, with modes increasing, the calculations will be more and more complicated.

Suppose

$$L = \{\pm\boldsymbol{K}_1,\ \pm\boldsymbol{K}_2,\ \pm\boldsymbol{K}_3,\ \pm\boldsymbol{K}_4,\ \pm\boldsymbol{K}_5,\ \pm\boldsymbol{K}_6,\ \pm\boldsymbol{K}_7,\ \pm\boldsymbol{K}_8,\ \pm\boldsymbol{K}_9\},$$

and take L as the set of vectors

$$\begin{aligned} &\boldsymbol{K}_1 = (1, 1), && \boldsymbol{K}_2 = (3, 0), && \boldsymbol{K}_3 = (2, -1), \\ &\boldsymbol{K}_4 = (1, 2), && \boldsymbol{K}_5 = (0, 1), && \boldsymbol{K}_6 = (1, 0), \\ &\boldsymbol{K}_7 = (-1, 2), && \boldsymbol{K}_8 = (2, 1), && \boldsymbol{K}_9 = (0, 3) \end{aligned}$$

and their opposites. When $\nu = 1$, choose $\boldsymbol{K}$ as $\boldsymbol{K}_i$ $(i = 1, \cdots, 9)$ respectively in (2.7) and make the following transform:

$$\begin{aligned} &r_{\boldsymbol{K}_1} = 2\sqrt{10}x_1, && r_{\boldsymbol{K}_2} = -2\sqrt{10}\mathrm{i}x_2, && r_{\boldsymbol{K}_3} = 2\sqrt{10}x_3, \\ &r_{\boldsymbol{K}_4} = 2\sqrt{10}\mathrm{i}x_4, && r_{\boldsymbol{K}_5} = 2\sqrt{10}x_5, && r_{\boldsymbol{K}_6} = 2\sqrt{10}\mathrm{i}x_6, \\ &r_{\boldsymbol{K}_7} = -2\sqrt{10}\mathrm{i}x_7, && r_{\boldsymbol{K}_8} = 2\sqrt{10}x_8, && r_{\boldsymbol{K}_9} = 2\sqrt{10}x_9. \end{aligned}$$

Taking the force acting on each mode, after calculating we find that when it acts on mode k_3 and k_7 respectively, a complicated behavior about chaos exhibited.