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Uniquely Strongly Clean Group Rings*

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Abstract: A ring R is called clean if every element is the sum of an idempotent and a unit, and R is called uniquely strongly clean (USC for short) if every element is uniquely the sum of an idempotent and a unit that commute. In this article, some conditions on a ring R and a group G such that RG is clean are given. It is also shown that if G is a locally finite group, then the group ring RG is USC if and only if R is USC, and G is a 2-group. The left uniquely exchange group ring, as a middle ring of the uniquely clean ring and the USC ring, does not possess this property, and so does the uniquely exchange group ring.

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1 Introduction

In this paper, R is an associative ring with identity 1. A ring R is called clean if every element is the sum of an idempotent and a unit. This definition first appeared in the paper by Nicholson[1] in 1977, in which it was also proved that clean rings are exchange rings, i.e., a ring R is exchange if and only if for any $x \in R$, there exists $e^2 = e \in R$ such that $e \in Rx$ and $1 - e \in R(1 - x)$. And the two concepts are equivalent for rings with all idempotents central. A ring R is called uniquely clean if each element has a unique representation as the sum of an idempotent and a unit. For instance, every boolean ring is uniquely clean, and a homomorphic image of a uniquely clean ring is uniquely clean. Uniquely clean rings were discussed in [2–4]. Nicholson and Zhou[3] proved that a ring R is uniquely clean if and only if R modulo its Jacobson radical $J(R)$ is boolean, idempotents lift modulo $J(R)$, and idempotents in R are central if and only if for every $a \in R$ there exists a unique idempotent $e \in R$ such that $e - a \in J(R)$. A ring R is called strongly clean if every element of R is the sum of an idempotent and a unit that commute. Strongly clean rings were introduced by Nicholson[5]. Recently, Chen *et al.*[6] raised a new concept about uniquenly strongly clean

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(USC for short) ring. They called a ring R USC if every element is uniquely the sum of an idempotent and a unit that commute. They also gave the equivalent condition for USC ring, that is, a ring R is USC if and only if for all $a \in R$ there exists a unique idempotent $e \in R$ such that $ea = ae$ and $e - a \in J(R)$. Nicholson and Zhou[3], Chen *et al.*[6] proved the following results which we can use in this paper:

(1) If R is uniquely clean , then $R/J(R)$ is boolean, and $2 \in J(R)$;

(2) If R is USC, then $R/J(R)$ is boolean, and $2 \in J(R)$.

We denote by RG the group ring of G over R. The augmentation mapping

$$\varepsilon : RG \to R$$

is given by

$$\varepsilon\Big(\sum_{g\in G} a_g g\Big) = \sum_{g\in G} a_g$$

and its kernel, denoted by $\Delta(G)$ (or by Δ_{RG}), is an ideal generated by $\{1 - g, g \in G\}$. If H is a subgroup of G, then εH denotes the right ideal of RG generated by $\{1 - h, h \in H\}$. If H is a normal subgroup of G, then εH is an ideal and $RG/\varepsilon H \cong R(G/H)$. If I is a right ideal of R, then IG denotes the elements of RG with coefficients in I, when I is an ideal so is IG, and $RG/IG \cong (R/I)G$. For further details see [7].

Three years ago, Chen *et al.*[8] raised a question: if R is a ring and G is a group, when is the group ring RG clean? Wang[9] studied the cleanness of group rings for a class of Abelian p-groups. But we know that $Z_{(3)}S_3$ is a clean group ring, where S_3 is not Abelian. This motivates us to look at the cleanness of group rings of Abelian or non-Abelian groups. In Section 2, some conditions on a ring R and a group G such that RG is clean are given. Moreover, in Sections 3 and 4 it is shown that if G is a locally finite group, then the group ring RG is USC if and only if R is USC, and G is a 2-group. The left uniquely exchange group ring, as a middle ring of the uniquely clean ring and the USC ring, does not possess this property, and so does the uniquely exchange group ring. We give an example to indicate this.

Throughout this paper, R denotes an associative ring with identity 1. As usual $J(R)$ denotes the Jacobson radical of the ring R and $U(R)$ the group of units in R. We write $T_n(R)$ for the ring of all upper triangular $n \times n$ matrices over the ring R. Let G denote a group. Then a group G is called a p-group if every element of G is a power of p, where p is a prime. Let S_n stand for the symmetric group of degree n. The ring of integers is denoted by Z, and we write Z_n for the ring of integers modulo n.

2 Clean Group Rings

A group G is called locally finite if every finitely generated subgroup is finite.

Lemma 2.1 *Let R be a ring, G a group, and $\triangle(G) \subseteq J(RG)$. Then*

$$J(RG) = \{\gamma \in RG \mid \varepsilon(\gamma) \in J(R)\}.$$