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Invertible Linear Maps on the General Linear Lie Algebras Preserving Solvability*

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Abstract: Let M_n be the algebra of all $n \times n$ complex matrices and $gl(n, \mathbb{C})$ be the general linear Lie algebra, where $n \geq 2$. An invertible linear map $\phi : gl(n, \mathbb{C}) \to gl(n, \mathbb{C})$ preserves solvability in both directions if both ϕ and ϕ^{-1} map every solvable Lie subalgebra of $gl(n, \mathbb{C})$ to some solvable Lie subalgebra. In this paper we classify the invertible linear maps preserving solvability on $gl(n, \mathbb{C})$ in both directions. As a sequence, such maps coincide with the invertible linear maps preserving commutativity on M_n in both directions.
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1 Introduction

Let $\mathcal{L}$ be a Lie algebra. Recall that the derived Lie algebra $\mathcal{L}^{(1)}$ of $\mathcal{L}$ is the Lie ideal $[\mathcal{L}, \mathcal{L}]$ spanned by all $[x, y]$, $x, y \in \mathcal{L}$. To each Lie algebra $\mathcal{L}$ we associated the derived series:

$$\mathcal{L} \supseteq \mathcal{L}^{(1)} \supseteq \mathcal{L}^{(2)} = (\mathcal{L}^{(1)})^{(1)} \supseteq \cdots .$$

The Lie algebra $\mathcal{L}$ is solvable if there exists a positive integer r such that $\mathcal{L}^{(r)} = \{0\}$. The set of all $n \times n$ complex matrices is denoted by M_n when considered as a set or a linear space or an algebra. If the linear space M_n is equipped with the Lie product

$$[\ \cdot\ ,\ \cdot\ ] : [A, B] = AB - BA,$$

then it becomes a general linear Lie algebra, denoted by $gl(n, \mathbb{C})$.

A lot of attention has been paid to linear preserver problem, which concerns the characterization of linear maps on matrix spaces or algebras that leave certain functions, subsets, relations, etc., invariant. The earliest paper on linear preserver problem dates back to 1897 (see [1]), and a great deal of effort has been devoted to the study of this type of question since

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then. One may consult the survey papers [2–4] for details. For linear or nonlinear preserver problem concerning linear Lie algebras we refer to the literature [5–12]. The author in [7] characterized the invertible linear maps on simple Lie algebras of linear types preserving zero Lie products. Radjavi and Semrl in [11] characterized the nonlinear maps which preserve solvability in both directions on the general linear Lie algebras and the special linear Lie algebras. In this article we determine the invertible linear maps preserving solvability on $gl(n, \mathbb{C})$ in both directions, where an invertible linear map $\phi : gl(n, \mathbb{C}) \to gl(n, \mathbb{C})$ is said to preserve solvability in both directions if for any solvable Lie algebra $\mathfrak{s} \subseteq gl(n, \mathbb{C})$, both $\phi(\mathfrak{s})$ and $\phi^{-1}(\mathfrak{s})$ are solvable Lie algebras of $gl(n, \mathbb{C})$. Now we state our main theorem:

Theorem 1.1 *Let $\phi : gl(n, \mathbb{C}) \to gl(n, \mathbb{C})$ be an invertible linear map. The following two conditions are equivalent*:

(1) *ϕ preserves solvability in both directions*;

(2) *There exists a non-zero scalar $\mu \in \mathbb{C}$, a linear functional f on $gl(n, \mathbb{C})$ with $f(I) \neq -\mu$ and an invertible matrix $S \in gl(n, \mathbb{C})$ such that either*

$$\phi(X) = \mu S X S^{-1} + f(X)I$$

for every $X \in gl(n, \mathbb{C})$, or

$$\phi(X) = \mu S X^t S^{-1} + f(X)I$$

for every $X \in gl(n, \mathbb{C})$, where X^t denotes the transpose of X.

The above result determines an explicit form of the linear invertible map preserving solvability described in Theorem 1.1 of [11]. In [12], the author proved that any bijective linear commutativity preserving map ϕ on M_n is also one of the above two standard maps. Thus we have the following corollary.

Corollary 1.1 *Let ϕ be an invertible linear map on $gl(n, \mathbb{C})$. Then the following conditions are equivalent*:

(1) *ϕ preserves solvability in both directions*;

(2) *ϕ preserves zero Lie products in both directions.*

Here we specify some notations for later use. We denote by I the identity matrix in $gl(n, \mathbb{C})$ and by E_{ij} the matrix in $gl(n, \mathbb{C})$ whose sole nonzero entry 1 is in the (i, j)-position. Let $\mathbb{C}I$ be the set $\{aI | a \in \mathbb{C}\}$ of all scalar matrices, H the set of all diagonal matrices in $gl(n, \mathbb{C})$, and $\mathbf{n}^+$ (resp., $\mathbf{n}^-$) the set of all strictly upper (resp., low) triangular matrices. Let $\mathcal{D}$ be the set of the diagonalizable matrices. Denote the one-dimensional vector space $\mathbb{C}E_{st}$ by $\mathcal{L}_{st}$ for any pair (s, t), $1 \leq s \neq t \leq n$. And denote $\mathbb{C}^* = \mathbb{C} - \{0\}$.

2 Certain Invertible Linear Maps Preserving Solvability

In this section, we construct certain invertible linear maps preserving solvability in both directions on $gl(n, \mathbb{C})$, which will be used to describe arbitrary invertible linear maps preserving solvability in both directions.