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The Third Initial-boundary Value Problem for a Class of Parabolic Monge-Ampère Equations*

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Abstract: For the more general parabolic Monge-Ampère equations defined by the operator $F(D^2u + \sigma(x))$, the existence and uniqueness of the admissible solution to the third initial-boundary value problem for the equation are established. A new structure condition which is used to get a priori estimate is established.
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1 Introduction and Statement of the Main Results

In this paper, we discuss the third initial-boundary value problem for parabolic Monge-Ampère equations

$$-\frac{\partial u}{\partial t} + F(D^2u + \sigma(x)) = f(x, t) \qquad \text{in } Q_T, \tag{1.1}$$

$$\alpha(x)\frac{\partial u}{\partial \nu} + u = \phi(x, t) \qquad \text{on } \partial\Omega \times [0, T], \tag{1.2}$$

$$u = \psi(x, 0) \qquad \text{on } \Omega \times \{t = 0\}, \tag{1.3}$$

where Ω is a bounded uniformly convex domain in $\mathbf{R}^n$,

$$Q_T = \Omega \times (0, T],$$
$$\partial_p Q_T = \partial\Omega \times (0, T] \cup \bar{\Omega} \times \{t = 0\},$$
$$F(D^2u + \sigma(x)) = \det{}^{\frac{1}{n}}(D^2u + \sigma(x)),$$

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and

$$D^2u = (D_{ij}u)$$

is the Hessian of u with respect to the variable x, ν is the unit exterior normal at $(x,t) \in \partial\Omega \times [0,T]$ to $\partial\Omega$, which has been extended on $\bar{Q}_T$ to be a properly smooth vector field independent of t, $\alpha(x) > 0$ is properly smooth for all $x \in \bar{\Omega}$, $\sigma(x) = (\sigma_{ij}(x))$ is an $n \times n$ symmetric matrix with smooth components, $f(x,t)$, $\phi(x,t)$, $\psi(x,t)$ are given properly smooth functions and satisfy some necessary compatibility conditions.

The first initial-boundary value problem for a class of elliptic Monge-Ampère equations

$$\begin{cases} \det(D^2u(x) + \sigma(x)) = f(x) & \text{in } \Omega, \\ u = \phi(x) & \text{on } \partial\Omega \end{cases}$$

was firstly discussed by Caffarelli *et al.*[1]

Ivochkina and Ladyzhenskaya[2] studied the following first initial-boundary value problem for parabolic Monge-Ampère equations

$$-\frac{\partial u}{\partial t} + \det{}^{\frac{1}{n}}(D^2u) = f(x,t) \qquad \text{in } Q_T, \tag{1.1*}$$

$$u = \phi(x,t) \qquad \text{on } \partial_pQ_T. \tag{1.2*}$$

They derived two structure conditions as follows:

$$\begin{cases} \min\limits_{\bar{Q}_T} f(x,t) + \min\limits_{\partial_pQ_T} \dfrac{\partial}{\partial t}\phi(x,0) - \dfrac{1}{2}ad^2 > 0, \text{ in which } d \text{ is} \\ \qquad \text{the radius of the minimal ball } B_d(x_0) \text{ containing } \Omega, \\ \qquad a = \max\left\{0,\ \max\limits_{\bar{Q}_T} \dfrac{\partial}{\partial t} f(x,t)\right\}, \end{cases} \tag{C$_2$)* ($\!$}$$

$$\begin{cases} \min\limits_{\partial_pQ_T} \left(f(x,t) + \dfrac{\partial}{\partial t}\phi(x,t)\right) > 0, \\ D^2f(x,t),\ D^2(\det{}^{\frac{1}{n}} D^2\phi(x,0)) \text{ are nonpositive definite.} \end{cases} \tag{C$'_2$)* ($\!$}$$

By $(\mathrm{C}_2)^*$ or $(\mathrm{C}'_2)^*$, they obtained the existence and uniqueness of the solution. The third initial-boundary value problem for equation $(1.1)^*$ was studied by Zhou and Lian[3]. They also got two structure conditions similar to $(\mathrm{C}_2)^*$ and $(\mathrm{C}'_2)^*$ in [2].

Therefore, it is natural for us to consider the problem (1.1)–(1.3) as an extension of the result of [2–3].

Definition 1.1 *We say that* $u(x,t)$ *is an admissible function of* (1.1)–(1.3) *if* $u(x,t) \in K$, *where*

$$K = \{v \in C^{2,1}(\bar{Q}_T) \mid (D^2v(x,t) + \sigma(x)) > 0,\ (x,t) \in \bar{Q}_T\}.$$

Definition 1.2 *We say that* $u(x,t)$ *is an admissible solution of* (1.1)–(1.3) *if an admissible function* $u(x,t)$ *satisfies* (1.1)–(1.3).

Obviously, the equation (1.1) is of parabolic type for any admissible function $u(x,t)$.

For any admissible solution, the following condition is necessary:

$$(D^2\psi(x,0) + \sigma(x)) > 0, \qquad x \in \bar{\Omega}. \tag{C$_1$)($\!$}$$