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Likely Limit Sets of a Class of p-order Feigenbaum's Maps*

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Abstract: A continuous map from a closed interval into itself is called a p-order Feigenbaum's map if it is a solution of the Feigenbaum's equation $f^p(\lambda x) = \lambda f(x)$. In this paper, we estimate Hausdorff dimensions of likely limit sets of some p-order Feigenbaum's maps. As an application, it is proved that for any $0 < t < 1$, there always exists a p-order Feigenbaum's map which has a likely limit set with Hausdorff dimension t. This generalizes some known results in the special case of $p = 2$.

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1 Introduction

This paper is concerned with the generalized Feigenbaum's equation

$$f^p(\lambda x) = \lambda f(x), \qquad p \geq 2, \tag{1.1}$$

where f is a continuous map of the closed interval $[0, 1]$ into itself, and f^p is the p-fold iteration of f. This equation was first studied in [1], and its original form is

$$f^2(\lambda x) = \lambda f(x),$$

which was posed by Feigenbaum [2−3] for explaining a universal phenomenon occurring in an interval mapping family with one parameter.

A continuous map f on the closed interval $[a, b]$ is said to be univallecular, if there exists $\alpha \in (a, b)$ such that f is strictly decreasing on $[a, \alpha]$ and strictly increasing on $[\alpha, b]$.

In the sequel, we use I to denote the interval $[0, 1]$.

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Definition 1.1 *Let f be a continuous map of I into itself. We call f a p-order Feigenbaum's map, if it is a solution of* (1.1)*, such that $f(0) = 1$ and $f|_{[\lambda,1]}$ is univallecular. We call Feigenbaum's map f non-univallecular, if f itself is not univallecular.*

In the past more than thirty years, the researches on Feigenbaum's maps have aroused one's grave concern (see [1–10]). More authors studied the existence of Feigenbaum's maps, and only a few of them were concerned with likely limit sets of Feigenbaum's maps. In [9], the Hausdorff dimension of a likely limit set of an even analytic function which is similar to Feigenbaum's map was estimated. It was pointed out in [10] that for any $t \in (0, 1)$, there is always a 2-order Feigenbaum's map which has a likely limit set with Hausdorff dimension t. As an extension of the results in [10], we study the likely limit sets of a class of p-order Feigenbaum's maps. The main results are given in Theorems 3.1 and 3.2.

2 Preliminaries

Milnor[11] introduced the concept of a likely limit set for a continuous map of a compact manifold as follows.

Let M be a compact manifold (possibly with boundary), and f be a continuous map from M into itself.

Definition 2.1 *The likely limit set of f, denoted by $\Lambda(f)$, or simply Λ, is the smallest closed invariant subset of M with the property $\omega(x, f) \subset \Lambda$ for each point $x \in M$ outside of a set of Lebesgue's measure zero, where $\omega(x, f)$ denotes the ω-limit set of the point x under f (see* [12] *for the definition).*

As indicated in [11], the likely limit set always exists and it is the unique maximal attractor (in the sense of Milnor). Because such an attractor accumulates the asymptotic behaviors of almost all points, it is very necessary to study this type of subsets.

Recall that a subset E of M is said to be minimal for f, if

$$E \neq \emptyset, \qquad \omega(x, f) = E, \quad x \in E.$$

As is well known, the minimal set is closed, non-void and invariant, and no proper subset has these three properties (see [12]). Therefore, if E is a minimal set with $\omega(x, f) \subset E$ for almost all $x \in M$, then

$$\Lambda(f) = E.$$

Let (X, d) be a compact metric space. Denote by $|E|$ the diameter of a subset E of X, i.e.,

$$|E| = \sup\{d(x, y) \mid x, y \in E\}.$$

Let $\delta > 0$. If $E \subset \bigcup\limits_i U_i$ and $0 < |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of E.

Let $E \subset X$, $0 \leq s < \infty$. For $\delta > 0$, define

$$\mathcal{H}^s_\delta(E) = \inf \sum_{i=1}^{\infty} |U_i|^s,$$