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Computing Numerical Singular Points of Plane Algebraic Curves*

Luo Zhong-xuan[1,2], Feng Er-bao[1] and Hu Wen-yu[1]
(*1. School of Mathematical Sciences, Dalian University of Technology, Dalian, Liaoning, 116024*)
(*2. School of Software, Dalian University of Technology, Dalian, Liaoning, 116620*)

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Abstract: Given an irreducible plane algebraic curve of degree $d \geq 3$, we compute its numerical singular points, determine their multiplicities, and count the number of distinct tangents at each to decide whether the singular points are ordinary. The numerical procedures rely on computing numerical solutions of polynomial systems by homotopy continuation method and a reliable method that calculates multiple roots of the univariate polynomials accurately using standard machine precision. It is completely different from the traditional symbolic computation and provides singular points and their related properties of some plane algebraic curves that the symbolic software Maple cannot work out. Without using multiprecision arithmetic, extensive numerical experiments show that our numerical procedures are accurate, efficient and robust, even if the coefficients of plane algebraic curves are inexact.
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1 Introduction

Singular points and their related properties play an important role in the theory of plane algebraic curves (see [1–2]), such as the computation of the genus of an algebraic curve. Practically, singular points present some shape features, such as nodes, self-intersections or cusps of real curves arising from computer aided geometry design, robot motion planning and machine vision. And computing singular points helps to determine the geometric shape and topology of the real curves.

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Some of the existing research in CAGD on computing singular points considers rational parametric curves. The computation of singular points of a rational parametric curve is studied in [3]. Chen *et al.*[3] surveyed other methods briefly therein.

Singular points of the plane algebraic curve $f(x,y)=0$ can be computed by solving the polynomial system

$$f(x,y)=f_x(x,y)=f_y(x,y)=0, \tag{1.1}$$

where $f_x(x,y)$ and $f_y(x,y)$ are the partial derivatives of $f(x,y)$ with respect to x and y respectively. Most of the existing methods in [4–8] solved this polynomial system either by resultant computation, which required the exact input of coefficients of algebraic curves, or by the Gröbner basis method described in [9]. For singular points of high multiplicity or algebraic curves with complex structures, the resultants may become difficult to compute. Furthermore, coefficients of algebraic curves obtained by fitting or interpolating the experimental data can seldom be exact. On the other hand, the reliance on symbolic manipulation of the Gröbner basis method makes the induced method seem somewhat limited to relatively small problem.

An algorithm can determine singular points of algebraic curves with rational coefficients, compute their multiplicities and count the number of distinct tangents at each using polynomial procedures (substitutions, resultants, greatest common divisors, etc.) in [10]. However, this algorithm gave only a real isolating interval or an isolating rectangle in the complex plane where every singular point lies, and no exact coordinates for every singular point.

For locating the singular points, Bajaj *et al.*[11] pointed out that a problem for their tracing curves algorithm is how to determine the singular points accurately. Our numerical procedures including the application of homotopy continuation method, provide the singular points quickly and accurately.

In this article, we compute the singular points indirectly by homotopy continuation method (see [12]), which is an efficient and reliable numerical algorithm for approximating all isolated zeros of polynomial systems. Different from [11] which solves the intersections of (1.1) using the Newton iteration, the homotopy continuation method provides all the isolated solutions of the polynomial systems globally. As we obtain the numerical solutions of the polynomial systems, we say numerical singular points corresponding to the singular points with exact coordinates obtained by resultant computations or the Gröbner basis method.

Singular points at infinity are computed by solving a univariate polynomial. It is different from the methods given in [5–6], as they all homogenized plane algebraic curves to the projective plane algebraic curves and solved other two overdetermined polynomial systems similar to (1.1) for computing singular points at infinity.

After all the numerical singular points are obtained, we calculate their multiplicities by examining derivatives of $f(x,y)$. The singular point is non-ordinary if there are multiple roots of the corresponding univariate polynomial. Recently, a reliable method that calculates multiple roots of the univariate polynomials accurately by using standard machine precision was developed by Zeng[13]. With the aid of this method, singular points at infinity and the related properties of all the singular points can be determined accurately.