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Solvability of Third-order Three-point Boundary Value Problems with Carathéodory Nonlinearity*

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Abstract: A class of third-order three-point boundary value problems is considered, where the nonlinear term is a Carathéodory function. By introducing a height function and considering the integration of this height function, an existence theorem of solution is proved when the limit growth function exists. The main tools are the Lebesgue dominated convergence theorem and the Schauder fixed point theorem.

Key words: nonlinear ordinary differential equation, multi-point boundary value problem, existence

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1 Introduction

Let $\frac{1}{2} \leq \eta < 1$. This paper deals with the following third-order three-point boundary value problem:

$$\text{(P)} \qquad \begin{cases} u'''(t) = f(t, u(t)), & \text{a.e. } t \in [0, 1], \\ u(0) = A, \quad u'(\eta) = B, \quad u''(1) = C. \end{cases}$$

The existence of the solution of (P) has been investigated by many authors when $A = B = C = 0$ and $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ is continuous; for example, see [1–8] and the references therein.

Our purpose is to establish an existence theorem of the solution to the problem (P) when $f(t, u)$ is a Carathéodory function and there exists a limit growth function $\lim\limits_{u\to\infty} \left|\frac{f(t, u)}{u}\right|$. Here, $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ is referred to as a Carathéodory function if

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(H1) For a.e. $t \in [0,1]$, $f(t,\ \cdot\ ) : \mathbf{R} \to \mathbf{R}$ is continuous;

(H2) For any $u \in R$, $f(\ \cdot\ , u) : [0,1] \to \mathbf{R}$ is measurable;

(H3) For each $r > 0$, there is a nonnegative function $j_r \in L^1[0,1]$ such that

$$|f(t,u)| \leq j_r(t), \qquad (t,u) \in [0,1] \times [-r,r].$$

Hence, we allow that there exists a measurable subset $e \subset [0,1]$ with $m(e) = 0$ such that the nonlinear term $f(t,u)$ is discontinuous or singular on $e \times \mathbf{R}$. Here, $m(e)$ is the Lebesgue measure of the set e.

In this paper, we improve and apply the localization method used in [6–13]. In order to describe the growth feature of the nonlinear term $f(t,u)$ on a bounded set, we introduce a height function. By considering the integration of this height function and applying the Schauder fixed point theorem, we prove a local existence theorem. In the proof, we need to take the limits under the integral sign. The Lebesgue dominated convergence theorem yields very effective tool. Main result shows that the problem (P) can have a solution if

$$\int_0^1 \lim_{u\to\infty} \Big|\frac{f(t,u)}{u}\Big| \mathrm{d}t < \frac{2}{\eta^2}.$$

2 Preliminaries

Consider the Banach space $C[0,1]$ with the norm

$$\|u\| = \max_{0\leq t\leq 1} |u(t)|.$$

Let

$$p(t) = \frac{1}{2}Ct^2 + (B - C\eta)t + A.$$

Then

$$p(0) = A, \qquad p'(\eta) = B, \qquad p''(1) = C.$$

Write

$$\gamma = \max_{0\leq t\leq 1} |p(t)|.$$

Let $G(t,s)$ be the Green's function of the homogeneous linear problem

$$\begin{cases} u'''(t) = 0, & 0 \leq t \leq 1, \\ u(0) = u'(\eta) = u''(1) = 0, \end{cases}$$

that is,

$$G(t,s) = \begin{cases} ts - \dfrac{1}{2}t^2, & 0 \leq s \leq \eta,\ 0 \leq t \leq s; \\ \dfrac{1}{2}s^2, & 0 \leq s \leq \eta,\ 0 \leq s < t; \\ \eta t - \dfrac{1}{2}t^2, & \eta \leq s \leq 1,\ 0 \leq t \leq s; \\ \dfrac{1}{2}s^2 - ts + \eta t, & \eta \leq s \leq 1,\ 0 \leq s < t. \end{cases}$$

Thus,

$$G(t,s) > 0, \qquad 0 < t, s < 1.$$