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Fourier Moment Method with Regularization for the Cauchy Problem of Helmholtz Equation*

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Abstract: In this paper, we consider the reconstruction of the wave field in a bounded domain. By choosing a special family of functions, the Cauchy problem can be transformed into a Fourier moment problem. This problem is ill-posed. We propose a regularization method for obtaining an approximate solution to the wave field on the unspecified boundary. We also give the convergence analysis and error estimate of the numerical algorithm. Finally, we present some numerical examples to show the effectiveness of this method.

Key words: Fourier moment method, Cauchy problem, Helmholtz equation, regularization, ill-possedness

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1 Introduction

Consider the reconstruction of the wave field in a bounded domain Ω from unexact Cauchy data given on an open part Γ of the boundary $\partial\Omega$. This problem is described by a Cauchy problem for the Helmholtz equation, which arises in many physical applications related to wave propagation and vibration phenomena. For example, the acoustic cavity problem, the vibration of a structure, the heat conduction in fins, the radiation and scattering of wave (see [1–5] and the references therein).

It is well known that the Cauchy problem for elliptic equations is ill-posed (see [6]). Although the solution of Cauchy problem for the Helmholtz equation is unique, it does not depend continuously on Cauchy data. It implies serious numerical difficulties in solving this problem. However, this case is important from point of view of practical applications for acoustic and electromagnetic fields where the exact Cauchy data is approximated by their measurements. Therefore, these are important studies in the literature of the Cauchy

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problem for the Helmholtz equation. A number of numerical methods have been proposed to solve this problem. For example, there are the acoustic intensity-based method (see [7–8]), the Fourier regularization method (see [9]), the alternating iterative boundary element method (see [10]), the conjugate gradient boundary element method (see [11]), the method of fundamental solution (see [12]), and the moment method (see [13]).

In this paper, a numerical method for reconstructing the wave field and its normal derivative on $\partial\Omega\backslash\Gamma$ is proposed. The main idea is to transform the Cauchy problem into a Fourier moment problem, whose solution has an expansion with respect to trigonometric polynomials. The wave field and its normal derivative on the unspecified boundary can be obtained by solving the corresponding Fourier moment problem. Then we only need to solve the boundary value problems for the Helmholtz equation, i.e. Dirichlet, Neumann or mixed boundary value problems (see [2, 14–17]).

The layout of the paper is as follows. In Section 2, we transform the Cauchy problem into a Fourier moment problem. In Section 3, the numerical method with regularization for solving the Fourier moment problem is proposed. The convergence analysis and error estimate are also presented. Then we give the numerical algorithm for solving the normal derivative of the wave field on $\partial\Omega\backslash\Gamma$ in this section. We present several numerical examples to illustrate the competitive behavior of the method in Section 4. Some conclusions are given in Section 5.

2 Formulation of the Fourier Moment Problem

Let Ω be a bounded and simply connected domain in $\mathbf{R}^2$, whose boundary $\partial\Omega$ is sufficiently smooth. Γ is an open part of the boundary $\partial\Omega$ in the half plane $\{(x,y) \mid y \geqslant 0\}$ which connects two points $(0,0)$ and $(1,0)$. $\partial\Omega\backslash\Gamma = \{(x,y) \mid y=0,\ 0\le x\le 1\}$ is supposed to be a special curve.

For $s\in\mathbf{R}$, we denote the Sobolev space on Γ by $H^s(\Gamma)$. Let $f\in H^{\frac{3}{2}}(\Gamma)$, $g\in H^{\frac{1}{2}}(\Gamma)$. Consider the following Cauchy problem:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u|_\Gamma = f, & \\ \left.\dfrac{\partial u}{\partial \nu}\right|_\Gamma = g, & \end{cases} \tag{2.1}$$

where $k>0$ is the wave number and ν is the unit outward normal to the boundary Γ. In this paper, we assume that $u(0,0)=u(1,0)$.

For $n\in\mathbf{Z}$, let

$$v_n(x,y) = \begin{cases} \dfrac{-1}{\eta_n}\sin(\eta_n y)\mathrm{e}^{-\mathrm{i}2n\pi x}, & |n| < \dfrac{k}{2\pi}; \\ -y\mathrm{e}^{-\mathrm{i}2n\pi x}, & |n| = \dfrac{k}{2\pi}; \\ \dfrac{-1}{\eta_n}\sinh(\eta_n y)\mathrm{e}^{-\mathrm{i}2n\pi x}, & |n| > \dfrac{k}{2\pi}, \end{cases} \tag{2.2}$$