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On Commuting Graph of Group Ring Z_nS_3*

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Abstract: The commuting graph of an arbitrary ring R, denoted by $\Gamma(R)$, is a graph whose vertices are all non-central elements of R, and two distinct vertices a and b are adjacent if and only if $ab = ba$. In this paper, we investigate the connectivity and the diameter of $\Gamma(Z_nS_3)$. We show that $\Gamma(Z_nS_3)$ is connected if and only if n is not a prime number. If $\Gamma(Z_nS_3)$ is connected then $\mathrm{diam}(\Gamma(Z_nS_3)) = 3$, while if $\Gamma(Z_nS_3)$ is disconnected then every connected component of $\Gamma(Z_nS_3)$ must be a complete graph with same size, and we completely determine the vertice set of every connected component.

Key words: group ring, commuting graph, connected component, diameter of a graph

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1 Introduction

Let G be a group and R a ring. We denote by RG the set of all formal linear combinations of the form

$$\alpha = \sum_{g\in G} a_g g,$$

where $a_g \in R$ and $a_g = 0$ almost everywhere, that is, only a finite number of coefficients are different from 0 in each of these sums. Notice that it follows from our definition that given two elements

$$\alpha = \sum_{g\in G} a_g g \in RG, \qquad \beta = \sum_{g\in G} b_g g \in RG,$$

we have that

$$\alpha = \beta$$

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if and only if

$$a_g = b_g, \qquad g \in G.$$

We define the sum of two elements in RG componentwise:

$$\Big(\sum_{g\in G} a_g g\Big) + \Big(\sum_{g\in G} b_g g\Big) = \sum_{g\in G}(a_g + b_g)g.$$

Also, given two elements

$$\alpha = \sum_{g\in G} a_g g \in RG, \qquad \beta = \sum_{h\in G} b_h h \in RG,$$

we define their product by

$$\alpha\beta = \sum_{g,\ h\in G} a_g b_h gh.$$

The commuting graph of an arbitrary ring R denoted by $\Gamma(R)$ is a graph with vertex set $V(R) = R \setminus \mathcal{Z}(R)$, where $\mathcal{Z}(R)$ is the center of R, and two distinct vertices a and b are adjacent if and only if $ab = ba$. The notion of commuting graph of a ring was first introduced by Akbari *et al.*[1] in 2004. They investigated some properties of $\Gamma(R)$, whenever R is a finite semisimple ring. For any finite field F, they obtained connectivity, minimum degree, maximum degree and clique number of $\Gamma(Mn(F))$. Also it was shown that for any two finite semisimple rings R and S, if $\Gamma(R) \cong \Gamma(S)$, then there are commutative semisimple rings R_1 and S_1 and semisimple ring T such that

$$R \cong T \times R_1, \qquad S \cong T \times S_1, \qquad |R_1| = |S_1|.$$

The commuting graphs of some special rings have also been studied (see [2–4]).

Group rings are very interesting algebraic structure. For a group ring Z_nS_3, the properties of commuting graph can reflect its some structures. In this paper, we investigate some properties of $\Gamma(Z_nS_3)$, where

$$Z_nS_3 = \{x_1 + x_2a + x_3a^2 + x_4b + x_5ab + x_6a^2b \mid x_i \in Z_n,\ i = 1, 2, \cdots, 6\},$$
$$S_3 = \langle a,\ b \mid a^3 = b^2 = 1,\ bab = a^{-1}\rangle = \{1,\ a,\ a^2,\ b,\ ab,\ a^2b\}$$

is the symmetric group of order 6, and

$$Z_n = \{0,\ 1,\ \cdots,\ n-1\}$$

is the module n residue class ring. Given a group ring RG and a finite subset X of the group G, we denote by $\widehat{X}$ the following element of RG:

$$\widehat{X} = \sum_{x\in X} x.$$

In addition, the distinct conjugacy classes of S_3 are

$$\mathscr{C}_1 = \{1\}, \qquad \mathscr{C}_2 = \{a,\ a^2\}, \qquad \mathscr{C}_3 = \{b,\ ab,\ a^2b\}.$$

By Theorem 3.6.2 in [5], $\{\widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3}\}$ form a basis of the center $\mathcal{Z}(Z_nS_3)$, where $\widehat{\mathscr{C}_i}$ denotes the class sum.

In this paper, all graphs are simple and undirected and $|G|$ denotes the number of vertices of the graph G. We write $x \in V(G)$ when x is a vertex of G. A path of length r from a vertex x to another vertex y in G is a sequence of $r + 1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. For a connected graph H,