Communications in Mathematical Research
28(4)(2012), 349–358

A Joint Density Function in Phase-type (2) Risk Model*

Xu Huai[1] and Tang Ling[2]

(*1. School of Mathematics, Anhui University, Hefei, 230039*)

(*2. Department of Mathematics, Anhui Institute of Architecture and Industry, Hefei, 230601*)

Communicated by Wang De-hui

Abstract: In this paper, we consider a Gerber-Shiu discounted penalty function in Sparre Andersen risk process in which claim inter-arrival times have a phase-type (2) distribution, a distribution with a density satisfying a second order linear differential equation. By conditioning on the time and the amount of the first claim, we derive a Laplace transform of the Gerber-Shiu discounted penalty function, and then we consider the joint density function of the surplus prior to ruin and the deficit at ruin and some ruin related problems. Finally, we give a numerical example to illustrate the application of the results.

Key words: Gerber-Shiu discounted penalty function, phase-type (2) distribution, surplus prior to ruin, deficit at ruin

2000 MR subject classification: 60P05, 60H10

Document code: A

Article ID: 1674-5647(2012)04-0349-10

1 Introduction

The Sparre Andersen surplus process $\{U(t)\}_{t\geq 0}$ is defined by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \tag{1.1}$$

where u is the initial surplus, c is the rate of premium income per unit time, $\sum\limits_{i=1}^{N(t)} X_i$ denotes the aggregate claim amount up to time t, X_i represents the amount of the ith claim, and $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables with distribution function

$$F(x) = 1 - \bar{F}(x) = P\{X_i \leq x\}$$

*Received date: Sept. 28, 2010.
Foundation item: Tianyuan NSF (11126176) of China.

and density function $f(x)$.

Denote the Laplace transform of $f(x)$ by

$$f^*(s) = E(\mathrm{e}^{-sX_i}) = \int_0^\infty \mathrm{e}^{-sx} f(x)\mathrm{d}x$$

(with similar notation for Laplace transforms of other functions). Let

$$\mu = E(X_i) = \int_0^\infty x f(x)\mathrm{d}x < \infty.$$

We introduce

$$q(x) = \frac{1 - F(x)}{\mu}$$

and its Laplace transform

$$q^*(s) = \frac{1 - f^*(s)}{\mu s}.$$

$\{N(t)\}_{t \geq 0}$ is a counting process with $N(t)$ denoting the number which claims up to time t. The sequence of i.i.d. random variables $\{W_i\}_{i=1}^\infty$ represents the claim inter-arrival times where W_1 is the time until the first claim. We assume that claim amounts are independent of claim inter-arrival times. W_i has distribution function

$$K(t) = 1 - \bar{K}(t) = P\{W_i \leq t\}$$

and density function $k(t)$. Furthermore, we assume that

$$cE(W_i) - E(X_i) = c\rho - \mu > 0,$$

where

$$\rho = E(W_i) = \int_0^\infty t k(t)\mathrm{d}t < \infty.$$

Let T denote the time of ruin. $|U_T|$ and U_{T-} denote the deficit at ruin and the surplus prior to ruin, respectively. Let

$$\phi(u) = E[w(U_{T-},\ |U_T|)I(T < \infty) \mid U(0) = u],$$

where $w(\ \cdot\ ,\ \cdot\ )$ is a non-negative function, and

$$I(A) = \begin{cases} 1, & \text{if } A \text{ occurs;} \\ 0, & \text{otherwise.} \end{cases}$$

We remark that our function $\phi(u)$ is a simple version of the more general Gerber-Shiu discounted penalty function

$$\varphi(u) = E[\mathrm{e}^{-\delta T} w(U_{T-},\ |U_T|)I(T < \infty) \mid U(0) = u].$$

We set $\delta = 0$ simply in this paper. The Gerber-Shiu function is well recognized in risk model. Landriault and Willmot[1] considered the Gerber-Shiu function in the Sparre Andersen model with an arbitrary interclaim time distribution. Zhao and Yin[2] studied the Gerber-Shiu expected discounted penalty function for Levy insurance risk process. Cheung *et al.*[3] considered some structural properties of Gerber-Shiu expected discounted penalty function in dependent Sparre Andersen model. By the Markov property of a joint process, Song *et al.*[4] proved the systems of integro-differential equations for the Gerber-Shiu functions and obtained closed form expressions for the Gerber-Shiu functions when the claim amount distribution is from the rational family.