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Extinction of Weak Solutions for Nonlinear Parabolic Equations with Nonstandard Growth Conditions*

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Abstract: This paper deals with the extinction of weak solutions of the initial and boundary value problem for $u_t = \mathrm{div}((|u|^\sigma + d_0)|\nabla u|^{p(x)-2}\nabla u)$. When the exponent belongs to different intervals, the solution has different singularity (vanishing in finite time).
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1 Introduction

Let $\Omega \subset \mathbf{R}^N$ $(N \geqslant 2)$ be a bounded Lipschitz domain and $0 < T < \infty$. Consider the following general quasilinear degenerate parabolic problem:

$$\begin{cases} u_t = \mathrm{div}((u^\sigma + d_0)|\nabla u|^{p(x)-2}\nabla u), & (x,\, t) \in Q_T, \\ u(x,\, t) = 0, & (x,\, t) \in \Gamma_T, \\ u(x,\, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $Q_T = \Omega \times (0,\, T]$ and Γ_T denotes the lateral boundary of the cylinder Q_T.

Assume throughout the paper that the exponent $p(x)$ is continuous in $\bar{\Omega}$ with logarithmic module of continuity:

$$1 < p^- = \inf_{x\in\Omega} p(x) \leqslant p(x) \leqslant p^+ = \sup_{x\in\Omega} p(x) < \infty, \tag{1.2}$$

$$|x - y| < 1, \quad |p(x) - p(y)| \leqslant \Omega(|x - y|), \qquad x, y \in \Omega, \tag{1.3}$$

where

$$\limsup_{\tau\to 0^+} \Omega(\tau) \ln \frac{1}{\tau} = C < +\infty.$$

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The model (1.1) proposed by Ružička[1] describe some properties of electro-rheological fluids which change their mechanical properties dramatically when an external electric field is applied. The variable exponent p in the model (1.1) is a function of the external electric field $|\boldsymbol{E}|^2$ which is subject to the quasi-static Maxwell's equations

$$\begin{cases}\operatorname{div}(\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = 0,\\ \operatorname{Curl}(\boldsymbol{E}) = 0,\end{cases}$$

where ε_0 is the dielectric constant in vacuum and the electric polarization $\boldsymbol{P}$ is linear in $\boldsymbol{E}$, i.e., $\boldsymbol{P} = \lambda \boldsymbol{E}$. For more physical backgrounds, the interested readers may refer to [2–4]. These models include parabolic or elliptic equations which are nonlinear with respect to gradient of the thought solution and with variable exponents of nonlinearity (see [5–8] and references therein). Besides, another important application is the image processing where the anisotropy and nonlinearity of the diffusion operator and convection terms are used to underline the borders of the distorted image and to eliminate the noise (see [9–11]).

In the case when p is a fixed constant, Yin and Jin[12] discussed the extinction and non-extinction of solutions by applying comparison theorem and energy estimate methods. However, we point out that these methods used in [12] fail in our problems. The main reason is that the following identities do not hold

$$\int_\Omega u^m |\nabla u|^{p(x)} \mathrm{d}x \not\equiv \int_\Omega \Big(\frac{p(\cdot)}{m+p(\cdot)}\Big)^{p(\cdot)} |\nabla u^{\frac{m+p(\cdot)}{p}}|^{p(x)} \mathrm{d}x,$$
$$\operatorname{div}(|\nabla(\lambda u)|^{p(x)-2}\nabla(\lambda u)) \not\equiv \lambda^{p(x)-1}\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

For many results about the existence, uniqueness, nonexistence and the properties of the solutions, we refer the readers to the bibliography given in [13–19].

To the best of our knowledge, there are only a few works about parabolic equations with variable exponents of nonlinearity. Applying Galerkin's method, Antontsev and Shmarev[4] obtained the existence and uniqueness of weak solutions with the assumption that the function $a(u)$ in $\operatorname{div}(a(u)|\nabla u|^{p(x)-2}\nabla u)$ is bounded. In the case when the function $a(u)$ in $\operatorname{div}(a(u)|\nabla u|^{p(x)-2}\nabla u)$ might be not upper bounded, Guo and Gao[20–21] applied the method of parabolic regularization and Galerkin's method to prove the existence of weak solutions. In this paper, we find when the exponent belongs to different intervals, the solution represents different singularity (vanishing in finite time). That is,

(1) If $\dfrac{2N}{N+2} \leqslant p^- < p^+ < 2$, the solution approaches 0 in L^2-norm as t goes to a finite time;

(2) If $\dfrac{2N}{N+4} < p^- < p^+ < \dfrac{2N}{N+2}$, the solution approaches 0 in L^r-norm $(r > 2)$ as t goes to a finite time;

(3) If $1 < p^- < \dfrac{2N}{N+4}$, $1 < p^+ < \dfrac{Np^-}{N-p^-}$, the solution approaches 0 in L^r-norm $(r > 2)$ as t goes to a finite time.

The outline of this paper is as follows: In Section 2, we introduce the function spaces of Orlicz-Sobolev type, and give the definition of the weak solution to the problem. Section 3 is devoted to the proof of the extinction of the solution obtained in Section 2.