Communications in Mathematical Research
29(1)(2013), 14–22

An Extended Multiple Hardy-Hilbert's Integral Inequality*

Hong Yong
(*Department of Mathematics, Guangdong University of Business Studies, Guangzhou, 510320*)

Communicated by Ji You-qing

Abstract: In this paper, by introducing the norm $\|\boldsymbol{x}\|_\alpha$ ($\boldsymbol{x} \in \mathbf{R}^n$), a multiple Hardy-Hilbert's integral inequality with the best constant factor and it's equivalent form are given.

Key words: multiple Hardy-Hilbert's integral inequality, weight function, best constant factor, β-function, Γ-function

2000 MR subject classification: 26D15

Document code: A

Article ID: 1674-5647(2013)01-0014-09

1 Introduction

If $p > 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $f \geq 0$, $g \geq 0$, and

$$0 < \int_0^\infty f^p(x)\mathrm{d}x < +\infty, \qquad 0 < \int_0^\infty g^q(x)\mathrm{d}x < +\infty,$$

then the well known Hardy-Hilbert's integral inequality is (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin\left(\dfrac{\pi}{p}\right)} \Big(\int_0^\infty f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}} \Big(\int_0^\infty g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}}. \tag{1.1}$$

Its equivalent form is

$$\int_0^\infty \Big(\int_0^\infty \frac{f(x)}{x+y}\mathrm{d}x\Big)^p \mathrm{d}y < \left[\frac{\pi}{\sin\left(\dfrac{\pi}{p}\right)}\right]^p \int_0^\infty f^p(x)\mathrm{d}x, \tag{1.2}$$

where the constant factors in (1.1) and (1.2) are optimal.

Hardy-Hilbert's inequality is important in harmonic analysis, real analysis and operator theory. In recent years, many valuable results (see [2–5]) have been obtained in generalization and improvement of Hardy-Hilbert's inequality. In 1999, Kuang[6] gave a generalization with a parameter λ of (1.1) as follows:

***Received date:** Jan. 26, 2010.

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\mathrm{d}y$$
$$< B^{\frac{1}{p}}\Big(\frac{1}{p},\ \lambda-\frac{1}{q}\Big)B^{\frac{1}{q}}\Big(\frac{1}{q},\ \lambda-\frac{1}{q}\Big)\Big(\int_0^\infty x^{1-\lambda}f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_0^\infty x^{1-\lambda}g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}}, \tag{1.3}$$

where $\max\Big\{\frac{1}{p},\ \frac{1}{q}\Big\}<\lambda\le 1$, $B(\ \cdot\ ,\ \cdot\ )$ is the β-function. Noticing that the constant factor in (1.3) is not optimal, and the range of values of λ is too narrow, in 2002, Yang[7] gave a new generalization of (1.3) as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\mathrm{d}y$$
$$< B\Big(\frac{p+\lambda-2}{p},\ \frac{q+\lambda-2}{q}\Big)\Big(\int_0^\infty x^{1-\lambda}f^p(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_0^\infty x^{1-\lambda}g^q(x)\mathrm{d}x\Big)^{\frac{1}{q}}, \tag{1.4}$$

where $\lambda>2-\min\{p,\ q\}$, and the constant factor $B\Big(\frac{p+\lambda-2}{p},\ \frac{q+\lambda-2}{q}\Big)$ is optimal.

At present, for multiple Hardy-Hilbert's integral inequality, many new results have be obtained (see [8–10]). In this paper, by the method of weight function, a higher-dimensional generalization of (1.4) is obtained, and its equivalent form is researched. For the sake of convenience, we introduce the following symbols:

$$\mathbf{R}_+^n=\{\boldsymbol{x}=(x_1,\ \cdots,\ x_n):\ x_1,\ \cdots,\ x_n>0\},$$
$$\|\boldsymbol{x}\|_\alpha=(x_1^\alpha+\cdots+x_n^\alpha)^{\frac{1}{\alpha}},\qquad \alpha>0.$$

Lemma 1.1[11] *If $p_i>0$, $a_i>0$, $\alpha_i>0$, $i=1,2,\cdots,n$, and $\Psi(u)$ is a measurable function, then*

$$\int\cdots\int_{x_1,\cdots,x_n>0;(\frac{x_1}{a_1})^{\alpha_1}+\cdots+(\frac{x_n}{a_n})^{\alpha_n}\le 1}\Psi\Big(\Big(\frac{x_1}{a_1}\Big)^{\alpha_1}+\cdots+\Big(\frac{x_n}{a_n}\Big)^{\alpha_n}\Big)x_1^{p_1-1}\cdots x_n^{p_n-1}\mathrm{d}x_1\cdots\mathrm{d}x_n$$
$$=\frac{a_1^{p_1}\cdots a_n^{p_n}\Gamma\Big(\frac{p_1}{\alpha_1}\Big)\cdots\Gamma\Big(\frac{p_n}{\alpha_n}\Big)}{\alpha_1\cdots\alpha_n\Gamma\Big(\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}\Big)}\int_0^1\Psi(u)u^{\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}-1}\mathrm{d}u. \tag{1.5}$$

Lemma 1.2 *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $n\in\mathbf{Z}_+$, $\alpha>0$, $\lambda>\max\{n(2-p),\ n(2-q)\}$, and set the weight function*

$$\omega_{\alpha,\lambda}(\boldsymbol{x},\ q)=\int_{\mathbf{R}_+^n}\frac{1}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{x}\|_\alpha)^\lambda}\Big(\frac{\|\boldsymbol{x}\|_\alpha}{\|\boldsymbol{y}\|_\alpha}\Big)^{\frac{2n-\lambda}{q}}\mathrm{d}\boldsymbol{y},$$

then

$$\omega_{\alpha,\lambda}(\boldsymbol{x},\ q)=\|\boldsymbol{x}\|_\alpha^{n-\lambda}\frac{\Gamma^n\Big(\frac{1}{\alpha}\Big)}{\alpha^{n-1}\Gamma\Big(\frac{1}{\alpha}\Big)}B\Big(\frac{n(q-2)+\lambda}{q},\ \frac{n(p-2)+\lambda}{p}\Big). \tag{1.6}$$

Proof. By (1.5) one has

$$\omega_{\alpha,\lambda}(\boldsymbol{x},\ q)=\|\boldsymbol{x}\|_\alpha^{\frac{2n-\lambda}{q}}\int_{\mathbf{R}_+^n}\frac{1}{(\|\boldsymbol{x}\|_\alpha+\|\boldsymbol{y}\|_\alpha)^\lambda}\|\boldsymbol{y}\|_\alpha^{\frac{\lambda-2n}{q}}\mathrm{d}\boldsymbol{y}$$