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$\mathcal{F}$-perfect Rings and Modules*

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Abstract: Let R be a ring, and let $(\mathcal{F}, \mathcal{C})$ be a cotorsion theory. In this article, the notion of $\mathcal{F}$-perfect rings is introduced as a nontrial generalization of perfect rings and A-perfect rings. A ring R is said to be right $\mathcal{F}$-perfect if F is projective relative to R for any $F \in \mathcal{F}$. We give some characterizations of $\mathcal{F}$-perfect rings. For example, we show that a ring R is right $\mathcal{F}$-perfect if and only if $\mathcal{F}$-covers of finitely generated modules are projective. Moreover, we define $\mathcal{F}$-perfect modules and investigate some properties of them.

Key words: $\mathcal{F}$-perfect ring, $\mathcal{F}$-cover, $\mathcal{F}$-perfect module, cotorsion theory, projective module
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1 Introduction

In 1953, Eckmann and Schopf[1] proved the existence of injective envelopes of modules over any associative ring. The dual problem, that is, the existence of projective covers was studied by Bass[2] in 1960. In spite of the existence of injective envelopes over any ring, he proved that over a ring R, all right modules have projective covers if and only if R is a right perfect ring. In [3], a ring R is called right almost-perfect if every flat right R-module is projective relative to R, and proved that a ring is right almost-perfect if and only if flat covers of finitely generated modules are projective. In this article, we introduce the concept of $\mathcal{F}$-perfect rings. We give some characterizations of $\mathcal{F}$-perfect rings. For example, we show that a ring R is right $\mathcal{F}$-perfect if and only if $\mathcal{F}$-covers of finitely generated modules are projective.

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Let $\mathcal{X}$ be a class of R-modules. We denote

$$^{\perp}\mathcal{X} = \ker \operatorname{Ext}^1(\,\cdot\,, X) = \{M \mid \operatorname{Ext}^1(M, X) = 0,\ \forall X \in \mathcal{X}\},$$
$$\mathcal{X}^{\perp} = \ker \operatorname{Ext}^1(X,\,\cdot\,) = \{N \mid \operatorname{Ext}^1(X, N) = 0,\ \forall X \in \mathcal{X}\}.$$

A pair $(\mathcal{F},\ \mathcal{C})$ of classes of R-modules is called a cotorsion theory if $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$ (see [4]). A cotorsion theory $(\mathcal{F},\ \mathcal{C})$ is called complete if every R-module has a special $\mathcal{C}$-preenvelope (and a special $\mathcal{F}$-precover) (see [5]). A cotorsion theory $(\mathcal{F},\ \mathcal{C})$ is called perfect if every R-module has a $\mathcal{C}$-envelope and an $\mathcal{F}$-cover (see [6, 7]). A cotorsion theory $(\mathcal{F},\ \mathcal{C})$ is said to be hereditary if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathcal{F}$ then L' is also in $\mathcal{F}$, or equivalently, if $0 \to C' \to C \to C'' \to 0$ is exact with $C', C \in \mathcal{C}$ then C'' is also in $\mathcal{C}$ (see [8]).

Let R be a ring and $\mathscr{C}$ be a class of R-modules which is closed under isomorphic copies. A $\mathscr{C}$-precover of an R-module M is a homomorphism $\varphi : F \to M$ with $F \in \mathscr{C}$ such that for any homomorphism $\psi : G \to M$ with $G \in \mathscr{C}$, there exists $\mu : G \to F$ such that $\varphi\mu = \psi$. A $\mathscr{C}$-precover $\varphi : F \to M$ is said to be a $\mathscr{C}$-cover if every endomorphism λ of F with $\varphi\lambda = \varphi$ is an automorphism of F. Dually, a $\mathscr{C}$-preenvelope and a $\mathscr{C}$-envelope of an R-module are defined.

In [4] a ring R is called right almost-perfect if every flat right R-module is projective relative to R; equivalently, flat covers of finitely generated right R-modules are projective. It was shown that right perfect rings are right almost-perfect, and right almost-perfect rings are semiperfect, but not conversely. In Section 2, we introduce the notion of $\mathcal{F}$-perfect rings as a generalization of the notion of almost-perfect rings, that is, we call a ring R $\mathcal{F}$-perfect in case F is projective relative to R for any $F \in \mathcal{F}$. We give some characterizations of $\mathcal{F}$-perfect rings. For example, in Theorem 2.1 we show that a ring R is right $\mathcal{F}$-perfect if and only if $\mathcal{F}$-covers of finitely generated modules are projective. And in Theorem 2.3 we prove that a ring R is right $\mathcal{F}$-perfect if and only if for every right R-modules F with $F \in \mathcal{F}$, if

$$F = P + U,$$

where P is a finitely generated projective summand of F and $U \leq F$, then

$$F = P \oplus V \qquad \text{for some } V \leq U.$$

In Section 3, we introduce the notion of $\mathscr{F}$-perfect modules, that is, let $(\mathcal{F},\ \mathcal{C})$ be a perfect cotorsion theory. We call an R-module M $\mathscr{F}$-perfect in case the $\mathscr{F}$-cover of every factor module of M is projective. We show that $\mathscr{F}$-perfectness is closed under factor modules, extensions, and finite direct sums. Also some characterizations of $\mathscr{F}$-perfect modules are given.

Throughout this article, all rings are associative with identity, and all modules are unitary right modules unless stated otherwise. For a ring R, let $J = J(R)$ be the Jacobson radical of R. $(\mathcal{F},\ \mathcal{C})$ denotes a cotorsion theory. $\mathcal{F}$ (resp., $\mathcal{C}$) denotes the $\mathcal{F}$ (resp., $\mathcal{C}$) of the cotorsion theory $(\mathcal{F},\ \mathcal{C})$ unless stated otherwise.

General background materials can be found in [4, 9–10].