COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
29(1)(2013), 68–87

The Étale Homology and the Cycle Maps in Adic Coefficients*

LI TING
(*Mathematical College, Sichuan University, Chengdu, 610064*)

Communicated by Du Xian-kun

Abstract: In this article, we define the ℓ-adic homology for a morphism of schemes satisfying certain finiteness conditions. This homology has these functors similar to the Chow groups: proper push-forward, flat pull-back, base change, cap-product, etc. In particular, on singular varieties, this kind of ℓ-adic homology behaves much better than the classical ℓ-adic cohomology. As an application, we give a much easier approach to construct the cycle maps for arbitrary algebraic schemes over fields. And we prove that these cycle maps kill the algebraic equivalences and commute with the Chern action of locally free sheaves.

Key words: ℓ-adic cohomology, cycle map, derived category
2000 MR subject classification: 14F20, 14C25
Document code: A
Article ID: 1674-5647(2013)01-0068-20

1 Introduction

The étale cohomology, especially the ℓ-adic cohomology, is one of the most important tools of modern algebraic and arithmetic geometry, which allows us to construct a "good" cohomology theory for varieties over fields of arbitrary characteristic. More specifically, people use the ℓ-adic cohomology $\mathrm{H}^*(X_{\text{ét}}, \mathbf{Z}_\ell)$ to substitute for singular cohomology on varieties of arbitrary characteristic. On a nonsingular varieties, the cohomology $\mathrm{H}^*(X_{\text{ét}}, \mathbf{Z}_\ell)$ has very good properties and produces rich results. But on singular varieties or more generally on arbitrary schemes, the cohomology $\mathrm{H}^*(X_{\text{ét}}, \mathbf{Z}_\ell)$ behaves not so good, and many important constructions and results are not valid. So on singular varieties, the étale homology is more suitable than the étale cohomology.

In this paper, we generalize the étale homology defined in [1] in the following three facets. First, we define the étale homology in adic coefficients, which we call the ℓ-adic homology. Secondly, our theory of ℓ-adic homology is defined over schemes separated and of

***Received date:** Nov. 25, 2010.
Foundation item: The NSF (10626036) of China.

finite type over base schemes satisfying certain finiteness conditions, not just the algebraic schemes over separably closed fields as in [1]. In particular, algebraic schemes over fields which are not necessarily separably closed, are considered by us. Since our theory is based on the adic formalism created by Ekedahl[2], the ℓ-adic homology over base schemes of certain finiteness conditions shares almost the same good functorial properties, with that over separably closed base fields. Thirdly, the ℓ-adic homology groups $\mathrm{H}_*(X, \mathscr{F})$ defined by us take value in arbitrary bounded complex $\mathscr{F}$, not just $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$ or $\mathbf{Z}/n\mathbf{Z}$ as in [1]. And almost all functors and properties are preserved when extending to complexes. Basing on this homology, we also extend the cycle maps defined in [1], from separably closed base fields, to arbitrary base fields of finite ℓ-adic cohomological dimension.

In Section 2, we briefly reiterate the category $\mathbf{D}_{\mathrm{c}}(X_{\text{ét}}, R_\bullet)$ together with the Grothendieck's six operations in [2].

In Section 3, we recite the properties of the functor $\mathbf{R}f^!$ and use the language of [2] to rewrite the trace morphisms introduced in [3, 4].

In Section 4, we define the ℓ-adic homology groups $\mathrm{H}_n(X/Y, \mathscr{N})$ and $\mathbb{H}_n(X/Y, \mathscr{N})$ for a morphism $X \to Y$ of schemes satisfying certain finiteness conditions. These homology groups behave similarly in many facets to the bivariant Chow groups $A^{-n}(X \to Y)$ defined in Ch. 17 of [5]. We define two maps: the push-forward maps f_* and the pull-back maps f^*, for the ℓ-adic homology groups, which correspond to these maps on Chow groups $\mathrm{CH}_*(X)$ defined in §1.4 and §1.7 of [5]; and most important, we prove that these two maps commute (see Theorem 4.1), which is essential to construct various cycle maps basing on ℓ-adic homology. Moreover, we define the base change maps on the ℓ-adic homology.

In Section 5, we apply the ℓ-adic homology in Section 4 to define the cycle map

$$\mathrm{cl}_{X,\ell} : \mathrm{CH}_*(X) \to \mathbb{H}_*(X, \mathbf{Z}_\ell)$$

for arbitrary algebraic scheme X over a field of finite cohomological dimension at ℓ. We prove that the cycle map $\mathrm{cl}_{X,\ell}$ commutes with the push-forward map f_* and the pull-back map f^*. And we prove that the cycle maps kill the algebraic equivalence of algebraic cycles.

In Section 6, we prove that the cycle map $\mathrm{cl}_{X,\ell}$ commutes with the Chow action $c_i(\mathscr{E}) \cap \bullet$ by locally free sheaves.

The following notations and conventions would be used.

Let $\mathbf{N}$ be the set of natural numbers, $\mathbf{Z}$ the domain of integers, and $\mathbf{Q}$ the field of the rational numbers. Let $\mathbf{Z}_\ell$ and $\mathbf{Q}_\ell$ be the ℓ-adic completions of $\mathbf{Z}$ and $\mathbf{Q}$ respectively.

A morphism $f\colon X \to Y$ of schemes is said to flat (resp. smooth) of relative dimension n if f is flat (resp. smooth) and all fibers of f are n-equidimensional.

A morphism $f\colon X \to Y$ of Noetherian schemes is said to be compactifiable if it factors as $f = \bar{f} \circ j$ where $j\colon X \hookrightarrow \bar{X}$ is an open immersion, and $\bar{f}\colon \bar{X} \to Y$ is a proper morphism. By Theorem 4.1 of [6], f is compactifiable if and only if it is separated and of finite type.

An algebraic scheme over a field k is a scheme separated, of finite type over k. A variety over k is an integral algebraic scheme over k.

If A is a Noetherian ring, we use $\mathbf{D}(A)$ to denote the derived category of A-modules, and define $\mathbf{D}_{\mathrm{fg}}(A)$ to be the full subcategory of $\mathbf{D}(A)$ consisting of complexes cohomologically