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PS-injective Modules, PS-flat Modules and PS-coherent Rings

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Abstract: A left ideal I of a ring R is small in case for every proper left ideal K of R, $K + I \neq R$. A ring R is called left PS-coherent if every principally small left ideal Ra is finitely presented. We develop, in this paper, PS-coherent rings as a generalization of P-coherent rings and J-coherent rings. To characterize PS-coherent rings, we first introduce PS-injective and PS-flat modules, and discuss the relation between them over some spacial rings. Some properties of left PS-coherent rings are also studied.

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1 Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. The Jacobson radical of R is denoted by $J(R)$ and its right singular ideal is denoted by $Z(R_R)$. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (resp. $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ (resp. $\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\mathrm{Tor}^R_n(M, N)$). The character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. If X is a subset of R, the right (resp. left) annihilator of X in R is denoted by $r_R(X)$ (resp. $l_R(X)$). For the usual notations we refer the reader to [1–4].

Let R be a ring. A left R-module M is called finitely presented if there is an exact sequence $0 \to K \to F \to M \to 0$, where F is finitely generated free and K is finitely generated. A ring R is called left coherent if every finitely generated left ideal of R is finitely presented. Coherent rings and their generalizations have been extensively studied by many authors (see [5–8]). A left ideal I of a ring R is small in case for every proper left ideal K of R, $K + I \neq R$. A ring R is said to be left J-coherent (see [5]) (resp. P-coherent (see [8]))

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if every finitely generated small (resp. principally) left ideal of R is finitely presented. In Section 3 of this paper, we say that a ring R is left PS-coherent if every principally small left ideal of R is finitely presented. The concept of PS-coherent rings is introduced as a proper generalization of J-coherent rings and P-coherent rings. Some examples of PS-coherent rings are given, and some properties of PS-coherent rings are studied. It is shown that R is left J-coherent if and only if the matrix ring $M_n(R)$ is left PS-coherent for all $n \geq 1$. If R is semiregular, then R is left P-coherent if and only if R is left PS-coherent.

In order to characterize PS-coherent rings, PS-flat modules are firstly introduced in Section 2. We discuss the relation between PS-injective modules and PS-flat modules over some spacial rings. In light of these facts, we give some characterizations of semiprimitive rings (that is, $J(R) = 0$).

Let $\mathcal{C}$ be the class of R-modules. For an R-module M, a homomorphism $g : C \to M$ with $C \in \mathcal{C}$ is called a $\mathcal{C}$-cover (see [2]) of M if the following hold:

(1) For any homomorphism $g' : C' \to M$ with $C' \in \mathcal{C}$, there exists a homomorphism $f : C' \to C$ with $g' = gf$;

(2) If f is an endomorphism of C with $gf = g$, then f must be an automorphism.

If (1) holds but (2) may not, $g : C \to M$ is called a $\mathcal{C}$-precover. Dually, we have the definition of a $\mathcal{C}$-(pre)envelope. $\mathcal{C}$-covers and $\mathcal{C}$-envelopes may not exist in general, but if they exist, they are unique up to isomorphism. In Section 3 of this paper, we show that R is left PS-coherent if and only if every right R-module has a PS-flat preenvelope. If R is left PS-coherent, then every left R-module has a PS-injective cover. Furthermore, we consider when every left R-module has an epimorphic PS-injective cover and when every right R-module has a monomorphic PS-flat preenvelope.

2 PS-injective Modules and PS-flat Modules

We start with the following definition.

Definition 2.1 *Let R be a ring and Ra be any principally small left ideal. A left R-module M is called PS-injective if every R-homomorphism $f : Ra \to M$ can be extended to $R \to M$, equivalently, if $\mathrm{Ext}^1(R/Ra, M) = 0$. A right R-module N is said to be PS-flat if the sequence $0 \to N \otimes Ra \to N \otimes R$ is exact, or equivalently, $\mathrm{Tor}_1(N, R/Ra) = 0$. Similarly, we have the concept of right PS-injective modules and left PS-flat modules.*

Remark 2.1 (1) A left R-module M is said to be divisible (see [8]) (or P-injective) if $\mathrm{Ext}^1(R/Ra, M) = 0$ for all $a \in R$. A right R-module N is called torsionfree if

$$\mathrm{Tor}_1(N, R/Ra) = 0, \qquad a \in R.$$

Clearly, every divisible module (resp. torsionfree module) is PS-injective (resp. PS-flat). But the converse is not true in general. For example, let $R = \mathbb{Z}$ be the ring of integers. Then every module is PS-injective and PS-flat because $J(R) = 0$. However, $\mathbb{Z}$ is not a divisible $\mathbb{Z}$-module and $\mathbb{Z}_2$ is not a torsionfree $\mathbb{Z}$-module.