COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
29(2)(2013), 131–142

Numerical Stability and Oscillations of Runge-Kutta Methods for Differential Equations with Piecewise Constant Arguments of Advanced Type

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Communicated by Ma Fu-ming

Abstract: For differential equations with piecewise constant arguments of advanced type, numerical stability and oscillations of Runge-Kutta methods are investigated. The necessary and sufficient conditions under which the numerical stability region contains the analytic stability region are given. The conditions of oscillations for the Runge-Kutta methods are obtained also. We prove that the Runge-Kutta methods preserve the oscillations of the analytic solution. Moreover, the relationship between stability and oscillations is discussed. Several numerical examples which confirm the results of our analysis are presented.

Key words: numerical solution, Runge-Kutta method, asymptotic stability, oscillation

2000 MR subject classification: 65L07, 65L20

Document code: A

Article ID: 1674-5647(2013)02-0131-12

1 Introduction

In this paper, we consider the differential equations with piecewise constant arguments (EPCA) of advanced type, given by

$$\begin{cases} u'(t) = au(t) + bu([t+1]), \qquad t \geq 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

Received date: April 26, 2011.
Foundation item: The NSF (11201084, 51008084) of China.
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where a, b, u_0 are real constants and $[\,\cdot\,]$ denotes the greatest integer function. The general form of (1.1) is

$$\begin{cases} u'(t) = f(t, u(t),\ u(\alpha(t))), & t \geq 0, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where the argument $\alpha(t)$ has intervals of constancy.

The theory of EPCA was initiated in [1–3]. In the literature, there are many papers dealing with the properties of EPCA, such as Wiener and Cooke[4], Xia *et al.*[5], Muroya[6] and Akhmet[7]. Significant parts of pioneer results for EPCA can be found in [8]. For more details of EPCA, the reader can see [9–11] and the references therein.

In recent years, much research focused on the numerical solutions of EPCA. The stability and the oscillations of numerical solutions of EPCA was investigated in [12–16]. As far as we know, very few results were obtained on combining the stability with the oscillations of the numerical solutions in the paper except for [17]. Different from [17], the novel idea of our paper is that we study both stability and oscillations of the numerical solutions by using the Runge-Kutta methods for the problem (1.1), and their relationships are analyzed quantitatively.

2 Preliminaries

In this section, we introduce some definitions and theorems which are useful for our paper.

Definition 2.1[8] *A solution of the problem* (1.1) *on* $[0, \infty)$ *is a function* $u(t)$ *which satisfies the conditions*:

(i) $u(t)$ *is continuous on* $[0, \infty)$;

(ii) *The derivative* $u'(t)$ *exists at each point* $t \in [0, \infty)$ *with the possible exception of the points* $[t] \in [0, \infty)$, *where one-sided derivatives exist*;

(iii) (1.1) *is satisfied on each interval* $[n, n+1) \subset [0, \infty)$ *with integral end-points.*

Theorem 2.1[8] *If* $b \neq \dfrac{a}{e^a - 1}$, *then the problem* (1.1) *has a unique solution on* $[0, \infty)$

$$u(t) = (m_0(\{t\}) + \lambda m_1(\{t\}))\lambda^{[t]} u_0, \tag{2.1}$$

where $\{t\}$ *is the fractional part of* t *and*

$$m_0(t) = e^{at}, \quad m_1(t) = (e^{at} - 1)a^{-1}b, \quad \lambda = \frac{b_0}{1 - b_1}, \quad b_0 = m_0(1), \quad b_1 = m_1(1).$$

Theorem 2.2[8] *The solution of the problem* (1.1) *is asymptotically stable for all* u_0, *if and only if*

$$(a + b)\left(b - \frac{a(e^a + 1)}{e^a - 1}\right) > 0. \tag{2.2}$$

Definition 2.2 *A non-trivial solution of the problem* (1.1) *is said to be oscillatory if there exists a sequence* $\{t_k\}_{k=1}^{\infty}$ *such that* $t_k \to \infty$ *as* $k \to \infty$ *and* $u(t_k)u(t_{k-1}) < 0$, *otherwise, it is called non-oscillatory. We say that the problem* (1.1) *is oscillatory if all the non-trivial solutions of* (1.1) *are oscillatory. We say that the problem* (1.1) *is non-oscillatory if all the non-trivial solutions of* (1.1) *are non-oscillatory.*