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# A Devaney Chaotic System Which Is Neither Distributively nor Topologically Chaotic

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**Abstract:** Weiss proved that Devaney chaos does not imply topological chaos and Oprocha pointed out that Devaney chaos does not imply distributional chaos. In this paper, by constructing a simple example which is Devaney chaotic but neither distributively nor topologically chaotic, we give a unified proof for the results of Weiss and Oprocha.

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## 1 Introduction

Devaney, distributional and topological chaos are a few of different versions of chaos. Let us first recall their concrete definitions.

Let $(X,\ d)$ be a metric space, and $f: X \to X$ continuous (sometimes $f$ is said to be a system). We call $f$ Devaney chaotic, briefly DevC, if it possesses the three properties as defined in [1]:

(1) transitivity, i.e., there exists a point $x \in X$ such that the orbit $\mathrm{orb}(x, f) = \{x,\ f(x),\ f^2(x),\ \cdots\}$ is dense in $X$;

(2) periodic density, i.e., the set of periodic points of $f$ is dense in $X$;

(3) sensitive dependence (on initial conditions).

To determine if a map is chaotic, it is sufficient to consider whether it possesses the transitivity and the periodic density, since the properties (1) and (2) in the definition of

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Devaney imply (3) for the case that $f$ is infinite (see [2–3]).

The notion of distributional chaos was given in [4] (where, however, distributional chaos is called "strong chaos"). We call $f$ distributively chaotic, briefly DC, if there exists an uncountable set $D \subset X$ such that any different points $x, y \in D$ form a distributively chaotic pair, briefly DC pair, i.e., there exists an $\varepsilon > 0$ such that

$$F_{xy}(\varepsilon) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, n, \varepsilon) = 0,$$

and for any $\varepsilon > 0$,

$$F^*_{xy}(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n}\xi(x, y, n, \varepsilon) = 1,$$

where

$$\xi(x, y, n, \varepsilon) = \#\{i \mid d(f^i(x), f^i(y)) < \varepsilon,\ 0 \le i \le n-1\}$$

(# denotes the cardinality).

The definition of topological entropy was introduced in [5]. For more detail discussion we refer the readers to [6]. In this note, the topological entropy of $f$ is denoted by ent($f$). $f$ is said to be topologically chaotic, briefly PTE, if ent($f$) > 0.

Many researchers gave their attention to the relations among Devaney, distributional and topological chaos (see [7–13]).

By the definitions, Devaney chaos is a global characteristic, but distributional and topological chaos are not. One can easily give an example which is either distributional or topological chaos but not Devaney chaos. However, the inverse implications are not so evident. In 1971, Weiss[7] found that the transitivity and the periodic density do not imply PTE, and he had proved essentially that DevC does not imply PTE. Recently, the conclusion of Weiss was restated in [8]. To show that DevC does not imply DC, Oprocha[9] constructed a Devaney chaotic subshift without DC pairs where. However, he did not give a strict proof.

In the present paper, by forming a simple example which is Devaney chaotic but neither distributively nor topologically chaotic, we give a unified proof for the results of Weiss and Oprocha.

## 2 Symbolic Space, Shift and Subshift

Let $S = \{0, 1\}$, $\Sigma = \{x = x_0x_1 \cdots \mid x_i \in S,\ i = 0, 1, 2, \cdots\}$, and define $\rho : \Sigma \times \Sigma \to R$ as: for any $x = x_0x_1 \cdots$, $y = y_0y_1 \cdots \in \Sigma$,

$$\rho(x, y) = \frac{1}{2^i},$$

where $i$ is the minimal integer such that $x_i \ne y_i$. It is not difficult to check that $\rho$ is a metric on $\Sigma$. $(\Sigma,\ \rho)$ is compact (see [6]) and called the one-sided symbolic space (with two symbols). Define $\sigma : \Sigma \to \Sigma$ by

$$\sigma(x) = x_1x_2 \cdots, \qquad x = x_0x_1 \cdots \in \Sigma.$$

$\sigma$ is continuous (see [6]) and is called the shift on $\Sigma$. If $X \subset \Sigma$ is closed and $\sigma(X) \subset X$, we call $(X,\ \sigma|_X)$ or $\sigma|_X$ a subshift of $\sigma$.