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Existence of Positive Solutions for Singular Fourth Order Coupled System with Sign Changing Nonlinear Terms

Xia Tian, Gao Jing-lu and Cao Chun-ling*
(*School of Mathematics, Jilin University, Changchun, 130012*)

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Abstract: This paper deals with the existence of positive solutions to a singular fourth order coupled system with integral boundary conditions. Since the nonlinear terms f, g may change sign or be singular at $t = 0$ or $t = 1$, the authors make a priori estimates to overcome some difficulties and apply Guo-Krasnoselskii fixed point theorem to prove the existence of solutions of the system under suitable assumptions. Finally, some examples to illustrate the main results are given.

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1 Introduction

In this paper, we investigate the existence of positive solutions to the following singular fourth order system of ordinary differential equations with integral boundary conditions:

$$\begin{cases} u^{(4)}(t) = \omega_1(t)f(t,\ u(t),\ v(t),\ u''(t),\ v''(t)), & 0 < t < 1, \\ v^{(4)}(t) = \omega_2(t)g(t,\ u(t),\ v(t),\ u''(t),\ v''(t)), & 0 < t < 1, \\ u(0) = u(1) = \displaystyle\int_0^1 g_1(s)u(s)\mathrm{d}s, \qquad u''(0) = u''(1) = \displaystyle\int_0^1 h_1(s)u''(s)\mathrm{d}s, \\ v(0) = v(1) = \displaystyle\int_0^1 g_2(s)v(s)\mathrm{d}s, \qquad v''(0) = v''(1) = \displaystyle\int_0^1 h_2(s)v''(s)\mathrm{d}s, \end{cases} \tag{1.1}$$

where $f,\ g,\ \omega_i,\ g_i(s)$ and $h_i(s)$, $i = 1,\ 2$, satisfy

(H_1) $f,\ g \in C[[0, 1] \times [0, +\infty) \times [0, +\infty) \times (-\infty, 0] \times (-\infty, 0], (-\infty, +\infty)]$;

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* **Corresponding author.**
E-mail address: 641829005@qq.com (Xia T), caocl@jlu.edu.cn (Cao C L).

(H$_2$) $\omega_i \in C[(0,1),[0,+\infty)]$, satisfies

$$0 < \int_0^1 \omega_i(s)\mathrm{d}s < +\infty,$$

and are probably singular at $t = 0$ or $t = 1$;

(H$_3$) $g_i,\ h_i \in L^1[0,1]$ are nonnegative, and $\mu_i,\ \nu_i \in (0,1)$, where

$$\mu_i = \int_0^1 g_i(s)\mathrm{d}s, \qquad \nu_i = \int_0^1 h_i(s)\mathrm{d}s.$$

In recent years, boundary value problems for the system of ordinary differential equations have been studied extensively. The main tools are fixed point theorems in cones for completely continuous operators, and the readers may refer to [1–5]. Lü *et al.*[5] considered the following problem:

$$\begin{cases} u^{(4)}(t) = f(t,\, v(t)), & (t,v) \in (0,1) \times \mathbf{R}^+, \\ -v''(t) = g(t,\, u(t)), & (t,u) \in (0,1) \times \mathbf{R}^+, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = 0. \end{cases} \tag{1.2}$$

By the fixed point theorem of cone expansion and compression, they obtained the existence of solutions to the system (1.2).

With regard to integral boundary value problems, many authors have studied the existence of solutions, and the interested readers may refer to [6–8]. Furthermore, a large amount of literature has been devoted to the study of the existence of positive solutions to boundary value problems in which the nonlinear functions are allowed to change sign. Ji *et al.*[9] obtained the existence of solutions for boundary value problem with sign changing nonlinearity by the fixed point theorem.

However, it seems that there are not much study of singular fourth order systems of ordinary differential equations with sign change on nonlinear terms. Motivated by the above works, we discuss the existence of positive solutions to the problem (1.1).

The main features of this paper are as follows: Firstly, the system (1.1) consists of two fourth order ordinary differential equations with integral boundary conditions. Secondly, the functions f and g depend on u, v, u'', v'', and the functions $\omega_1,\ \omega_2$ are allowed to be singular at $t = 0$ or $t = 1$. Moreover, the nonlinear terms $f,\ g$ are allowed to change sign. By making a priori estimates and calculating accurately we overcome some difficulties and apply Guo-Krasnoselskii fixed point theorem to prove the existence of solutions by choosing suitable function class to which the solutions belong.

The outline of this paper is as follows: In Section 2, we give some properties of the Green's function associated with the problem (1.1) and some necessary preliminaries. Section 3 is devoted to the proof of the existence of solutions to the problem (1.1). In Section 4, we give some examples to illustrate how the main results can be used in practice.

2 Preliminaries

In this section, we give some preliminaries and lemmas.