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A Family of Fifth-order Iterative Methods for Solving Nonlinear Equations

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Abstract: In this paper, we present and analyze a family of fifth-order iterative methods free from second derivative for solving nonlinear equations. It is established that the family of iterative methods has convergence order five. Numerical examples show that the new methods are comparable with the well known existing methods and give better results in many aspects.

Key words: Newton's method, iterative method, nonlinear equation, order of convergence

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1 Introduction

In this paper, we consider the iterative methods to find a simple root α of a nonlinear equation

$$f(x) = 0. \tag{1.1}$$

i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, where $f : I \subset \mathbf{R} \to \mathbf{R}$ for an open interval I is a scalar function. Newton's method is an important and basic approach for solving nonlinear equations (see [1]), and its formulation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.2}$$

This method converges quadratically. To increase the order of convergence of the iterative methods, many authors have developed new methods (see [2–11]).

A two-step predictor-corrector Householder method (see [12]) is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1.3}$$

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$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)f''(y_n)}{2f'^3(y_n)}. \tag{1.4}$$

It is observed that the method depends on second derivative, so its practical utility is restricted rigorously. Therefore, it is important and interesting to develop iterative methods which are free from second derivative and whose order is higher if possible. This is main motivation of this paper.

2 Development of Methods and Convergence Analysis

Let us consider approximating the equation (1.1) around the point $(x_n,\ f(x_n))$ by the equation

$$g(x) = ax^3 + bx^2 + cx + d. \tag{2.1}$$

We impose the condition

$$g''(x_n) = f''(x_n) \tag{2.2}$$

on (2.1). From (2.1)-(2.2) we get the value of b easily determined in terms of a:

$$2b = f''(x_n) - 6ax_n. \tag{2.3}$$

Then

$$g''(x) = 6ax + f''(x_n) - 6ax_n. \tag{2.4}$$

By (1.3) and (2.4), we have

$$f''(y_n) \approx g''(y_n) = f''(x_n) - \frac{6af(x_n)}{f'(x_n)}. \tag{2.5}$$

We consider

$$f''(x_n) \simeq \frac{f'(y_n) - f'(x_n)}{y_n - x_n}. \tag{2.6}$$

Combining (1.3)-(1.4) and (2.5)-(2.6), we obtain the following new family iterative method for solving (1.1).

Algorithm 2.1

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.7}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)(f'^3(x_n) - f'(y_n)f'^2(x_n) - \mu f^2(x_n))}{2f'^3(y_n)f'(x_n)f(x_n)}. \tag{2.8}$$

Theorem 2.1 *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subset \mathbf{R} \to \mathbf{R}$ for an open interval I. If x_0 is sufficiently close to α, then Algorithm* 2.1 *has fifth-order convergence.*

Proof. Let

$$e_n = x_n - \alpha, \qquad c_k = \frac{1}{k!} \cdot \frac{f^{(k)}(\alpha)}{f'(\alpha)}.$$

We use Taylor expansions as follows:

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)], \tag{2.9}$$