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Existence and Uniqueness of Weak Solutions to the p-biharmonic Parabolic Equation

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Abstract: We consider an initial-boundary value problem for a p-biharmonic parabolic equation. Under some assumptions on the initial value, we construct approximate solutions by the discrete-time method. By means of uniform estimates on solutions of the time-difference equations, we establish the existence of weak solutions, and also discuss the uniqueness.

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1 Introduction

Suppose that $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary. Let λ, p and q be positive numbers with $p > 2$ and $q > 2$. In this paper, we consider the following p-biharmonic parabolic initial-boundary value problem:

$$\frac{\partial u}{\partial t} + \Delta(|\Delta u|^{p-2}\Delta u) - \operatorname{div}(|\nabla u|^{q-2}\nabla u) + \lambda|u|^{q-2}u = 0, \qquad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u = \Delta u = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $\Delta(|\Delta u|^{p-2}\Delta u)$ is called a p-biharmonic operator.

When $p = 2$ and $\lambda = 0$, (1.1) becomes

$$\frac{\partial u}{\partial t} + \Delta^2 u - \operatorname{div}(|\nabla u|^{q-2}\nabla u) = 0. \tag{1.4}$$

It arises in epitaxial growth of nanoscale thin film (see [1–3]), where $u(x, t)$ denotes the height from the surface of the film in epitaxial growth, $\Delta^2 u$ denotes the capillarity-driven surface diffusion, and $\operatorname{div}(|\nabla u|^{q-2}\nabla u)$ denotes the upward hopping of atoms.

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Liu and Du[4] studied (1.4) relying on some necessary uniform estimates of the approximate solutions, and they proved the existence of radial symmetric solutions to (1.4) in the two-dimensional space.

When $p = q = 2$ and $\lambda = 0$, (1.1) becomes a particular case of the following equation in 1-dimension:

$$\frac{\partial u}{\partial t} = -[u^\gamma u_{xxx}]_x + [u^\mu u_x]_x \tag{1.5}$$

with $\gamma = \mu = 0$. It follows from a small-slope approximation to metal surface evolution, with surface-diffusion and evaporation-condensation represented by fourth-order and second-order diffusion terms, respectively (see [5–6]). (1.5) with $\gamma = 0$ can be considered as a semilinear limit of the classical Cahn-Hilliard model of phase separation, which is also widely studied (see [7–9] and the references therein).

(1.1) is a typical higher order equation. Because of the degeneracy, the problem (1.1)–(1.3) does not admit classical solutions in general. So, we introduce weak solutions in the sense of the following definition:

Definition 1.1 *A function u is said to be a weak solution of the problem* (1.1)–(1.3) *if the following conditions are satisfied*:

(1) $u \in L^\infty(0, T; W_0^{2,p}(\Omega)) \cap C(0, T; L^2(\Omega) \cap H^1(\Omega))$, $\dfrac{\partial u}{\partial t} \in L^\infty(0, T; W^{-2,p'}(\Omega))$, *where p' is the conjugate exponent of p and $W^{-2,p'}(\Omega)$ is the dual space of $W_0^{2,p}(\Omega)$*;

(2) *For any $\varphi \in C_0^\infty(Q_T)$, $Q_T = \Omega \times (0, T)$, the following integral equality holds*:

$$-\iint_{Q_T} u\frac{\partial \varphi}{\partial t}\,\mathrm{d}x\mathrm{d}t + \iint_{Q_T} |\Delta u|^{p-2}\Delta u \Delta\varphi \mathrm{d}x\mathrm{d}t - \iint_{Q_T} |\nabla u|^{q-2}\nabla u \nabla\varphi \mathrm{d}x\mathrm{d}t$$
$$+ \lambda \iint_{Q_T} |u|^{q-2}u\varphi \mathrm{d}x\mathrm{d}t = 0;$$

(3) $u(x, 0) = u_0(x)$.

In this paper, we discuss the existence of weak solutions in Section 2. Our method is based on the discrete-time method to construct approximate solutions. By means of uniform estimates on solutions of the time-difference equations, we prove the existence of weak solutions of the problem (1.1)–(1.3). Later on, we discuss the uniqueness of weak solutions in Section 3.

2 Existence of Weak Solutions

Theorem 2.1 *Let $u_0 \in W_0^{2,p}(\Omega)$. Then the problem* (1.1)–(1.3) *admits at least one weak solution.*

To prove this theorem, we first consider the following discrete-time problem:

$$\frac{1}{h}(u_{k+1}-u_k)+\Delta(|\Delta u_{k+1}|^{p-2}\Delta u_{k+1})-\mathrm{div}(|\nabla u_{k+1}|^{q-2}\nabla u_{k+1})+\lambda|u_{k+1}|^{q-2}u_{k+1} = 0, \tag{2.1}$$

$$u_{k+1}|_{\partial\Omega} = \Delta u_{k+1}|_{\partial\Omega} = 0, \qquad k = 0, 1, \cdots, N-1, \tag{2.2}$$

where $h = \dfrac{T}{N}$, and u_0 is the initial value.