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Stability of Cubic Functional Equation in Three Variables

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Abstract: In this paper, we prove a generalization of Hyers' theorem on the stability of approximately additive mapping and a generalization of Badora's theorem on approximate ring homomorphism. We also obtain more general stability theorem, which gives stability theorems on Jordan and Lie homomorphisms. The proofs of the theorems in this paper are given following essentially the Hyers-Rassias approach to the stability of the functional equations connected with Ulam's problem.
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1 Introduction

In 1960, Ulam[1] raised the following question concerning the stability of homomorphisms:

Ulam's Question Let G_1 be a group and G_2 be a metric group with a metric $d(\,\cdot\,,\,\cdot\,)$. Given any $\varepsilon > 0$, it exists a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies
$$d(f(xy),\ f(x)f(y)) \leq \delta, \qquad x, y \in G_1,$$
then there is a homomorphism $g : G_1 \to G_2$ with
$$d(f(x),\ g(x)) \leq \varepsilon, \qquad x \in G_1.$$

The first result in this direction is proved by Hyers[2] which established the stability of a group homomorphism.

Theorem 1.1[2] *Let $\varepsilon \geq 0$ and f be a function defined on an Abelian group $(G,\ +)$ with values in a Banach space $(Y,\ \|\cdot\|)$ satisfying*
$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon, \qquad x, y \in G.$$

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Then there exists a unique additive mapping $h : G \to Y$, *such that*

$$\|f(x) - h(x)\| \le \varepsilon, \qquad x \in G.$$

In 1978, Rassias[3] provided the following drastic generalization of Hyers's result which allows the Cauchy difference to be unbounded.

Theorem 1.2[3] *Consider* E_1, E_2 *to be two Banach spaces, and let* $f : E_1 \to E_2$ *be a mapping such that* $f(tx)$ *is continuous in* t *for each fixed* x. *Assume that there exist* $\theta \ge 0$ *and* $p \in [0, 1)$ *such that*

$$\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \le \theta, \qquad x, y \in E_1.$$

Then there exists a unique linear mapping $T : E_1 \to E_2$ *such that*

$$\frac{\|f(x) - T(x)\|}{\|x\|^p} \le \frac{2\theta}{2 - 2^p}, \qquad x \in E_1.$$

Badora[4] proved the following result concerning the stability of a ring homomorphism:

Theorem 1.3[4] *Let* $\mathcal{R}$ *be a ring,* $\mathcal{B}$ *be a Banach algebra, and* $\varepsilon, \delta \ge 0$. *Assume that* $f : \mathcal{R} \to \mathcal{B}$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon, \quad \|f(x \cdot y) - f(x)f(y)\| \le \delta, \qquad x, y \in \mathcal{R}.$$

Then there exists a unique ring homomorphism $h : \mathcal{R} \to \mathcal{B}$ *such that*

$$\|f(x) - h(x)\| \le \varepsilon, \qquad x \in \mathcal{R}.$$

During the last decades, Hyers' theorem has been generalized in various directions (see [5–13]). In this paper, we generalize Hyers' theorem and Badora's theorem above. Based on these, we give a stability theorem both on Jordan homomorphism and on Lie homomorphism.

2 Stability of Approximate Group Homomorphisms

We first prove a theorem on stability of approximate group homomorphisms, which generalizes Hyers' theorem.

Jun and Kim[14] introduced the following functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{2.1}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability problem for (2.1).

Park and Jung[15] introduced the functional equation

$$f(3x+y) + f(3x-y) = 3f(x+y) + 3f(x-y) + 48f(x), \tag{2.2}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability problem for (2.2).

Later, Najati and Moradlou introduced two functional equations (see [16–17])

$$f(ax+y) + f(ax-y) = af(x+y) + af(x-y) + 2a(a^2-1)f(x), \tag{2.3}$$

$$3f(x+3y) + f(3x-y) = 15f(x+y) + 15f(x-y) + 80f(y), \tag{2.4}$$