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A Gorni-Zampieri Pair of a Homogeneous Polynomial Map

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Abstract: In this paper, we improve the algorithm and rewrite the function makePairing for computing a Gorni-Zampieri pair of a homogeneous polynomial map. As an application, some counterexamples to PLDP (dependence problem for power linear maps) are obtained, including one in the lowest dimension ($n = 48$) in all such counterexamples one has found up to now.

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1 Introduction

Let $\boldsymbol{F} = (F_1, \cdots, F_n) : \mathbf{C}^n \to \mathbf{C}^n$ be a polynomial map and $J\boldsymbol{F}$ the Jacobian matrix of $\boldsymbol{F}$. The famous Jacobian Conjecture asserts that $\boldsymbol{F}$ is invertible if $\det J\boldsymbol{F} \in \mathbf{C}^*$; see Introduction of [1] or [2]. A polynomial map of the form $\boldsymbol{F} = \boldsymbol{X} + \boldsymbol{H}$ is called homogeneous of degree d if each H_i is homogeneous of degree d. And a polynomial map of the form

$$\boldsymbol{F_A} = \Big(X_1 + \Big(\sum_{j=1}^{n} a_{1j}X_j\Big)^d, \ \cdots, \ X_n + \Big(\sum_{j=1}^{n} a_{nj}X_j\Big)^d\Big)$$

is called power linear of degree d, where $\boldsymbol{A} = (a_{ij})$. In 1982, Bass *et al.*[2] showed that it suffices to investigate the Jacobian Conjecture for all cubic homogeneous maps, and in 1983, Drużkowski[3] showed that it suffices to consider all cubic linear maps.

Gorni and Zampieri[4] introduced a notion of "pairing" between homogeneous maps and power linear maps (see Definition 2.1 below). The interest of the concept comes from that it

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establishes connections between homogeneous maps and power linear maps, and preserves some important properties. In fact, a homogeneous map is invertible (tame, triangularizable, respectively) if and only if so is its Gorni-Zampieri pair (G-Z pair, for short) $\boldsymbol{F_A}$; see Section 6.2 of [5] for details and more such properties.

Due to the "pairing", one may study the properties of a homogeneous map through its G-Z pair $\boldsymbol{F_A}$, and vice versa. For example, it was shown through G-Z pairs in [4], that CLLC (Meisters' cubic linear linearization conjecture) is equivalent to the DMZ-conjecture for cubic homogeneous maps, and then by use of this result, a counterexample to CLLC was obtained by Van den Essen in [6]. Another example is the investigation of HDP (homogeneous dependence problem) and PLDP (dependence problem for power linear maps), which arise in the research of the Jacobian conjecture. HDP was proposed by several authors in 1990s, for example, see [7–8] or Section 7.1 of [1], which asks that, for a homogeneous map $\boldsymbol{F}=\boldsymbol{X}+\boldsymbol{H}$ with $J\boldsymbol{H}$ nilpotent, whether $H_1,\cdots,H_n$ are linearly dependent over $\mathbf{C}$? And PLDP asks if HDP has an affirmative answer for power linear maps, for example, see [9] or Chapter 6 of [5]. Counterexamples to HDP(n,d) (where n and d denote the dimension and degree respectively) were given by De Bondt for every $n\geq 5$ with suitable d, including the ones for every $n\geq 10$ with $d=3$ and $n\geq 6$ with $d=4$, see [10] or Chapter 4 of [5]. Counterexamples to PLDP were found by De Bondt for $n=53$, $d=3$, see Section 6.1 of [5], and independently by the authors in [9] for $n=67$, $d=3$. And in [9], connections between HDP and PLDP were established through G-Z pairs, and an algorithm was given to compute counterexamples to PLDP from those to HDP.

The G-Z pair is a useful tool in the research of some problems concerning polynomial maps. According to the construction for G-Z pairs in the original paper of Gorni and Zampieri[4], a function makePairing was given by Gorni in [11] for computing a G-Z pair of a homogeneous map. Unfortunately, the function makePairing is only valid for degree 3, and more crucially it may run into bugs sometimes, so there is a built-in check that should warn if the result is wrong.

In Section 2, we improve the algorithm and rewrite the function makePairing, such that it can not only guarantee the correctness but also be valid for degree 4. As an application, in Section 3, we give a function pldpExample (the function makePairing is called in this function) for computing counterexamples to PLDP from those to HDP, and by use of which some counterexamples to PLDP are obtained including one in the lowest dimension ($n=48$) in all counterexamples to PLDP one has found up to now. All the codes (written in Mathematica 5.1) are given in Section 4.

2 G-Z Pairs

In what follows, we always view the vectors in $\mathbf{C}^n$ as column vectors, and write a polynomial map as $\boldsymbol{F}=(F_1,\cdots,F_n)^{\mathrm{T}}$. Then a power linear map

$$\boldsymbol{F_A}=\Big(X_1+\Big(\sum_{j=1}^{n}a_{1j}X_j\Big)^d,\cdots,X_n+\Big(\sum_{j=1}^{n}a_{nj}X_j\Big)^d\Big)$$