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Annulus and Disk Complex Is Contractible and Quasi-convex

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Abstract: The annulus and disk complex is defined and researched. Especially, we prove that this complex is contractible and quasi-convex in the curve complex.

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1 Introduction

Let S be a closed orientable surface with genus at least 2. Harvey[1] defined the curve complex of S as follows. The curve complex of S is the complex whose vertices are the isotopy classes of essential simple closed curves on S, and $k+1$ vertices in the curve complex span a k-simplex if they are represented by pairwise disjoint curves. We denote the curve complex of S by $\mathcal{C}(S)$. Harer[2] proved that $\mathcal{C}(S)$ is homotopy equivalent to a bouquet of spheres of dimension $-\chi(S)$.

If S is a boundary component of an irreducible 3-manifold M, then we can define the disk complex $\varDelta(M, S)$ as in [3]. A vertex of $\varDelta(M, S)$ is an isotopy class of an essential curve in S which bounds a disk in M. As in the definition of $\mathcal{C}(S)$, $k+1$ vertices in $\varDelta(M, S)$ span a k-simplex if they are represented by pairwise disjoint curves. It is easy to see that $\varDelta(M, S)$ is a subcomplex of $\mathcal{C}(S)$. McCullough[3] researched this complex and proved that it is contractible.

In Section 2, we define a new complex associated to a compression body as a generalization of both curve complex and disk complex of a handlebody. For a compression body C, we denote this new complex by $\mathcal{AD}(C)$ and call it annulus and disk complex. By using the techniques in [3], we prove the following theorem:

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Theorem 1.1 *The annulus and disk complex $\mathcal{AD}(C)$ is contractible.*

A metric space (X, d) is geodesic, if for any pair of points there is a path connecting them which is a geodesic; and a subset Y of (X, d) is K-quasi-convex if for any pair of points in Y, any geodesic in X connecting them lies in a K-neighborhood of Y. A result in [4] implies that $\Delta(M, S)$ is quasi-convex in $\mathcal{C}(S)$. By the aid of their results, we prove

Theorem 1.2 *$\mathcal{AD}(C)$ is K-quasi-convex in $\mathcal{C}(S)$, where K depends only on the genus of S.*

2 Preliminaries

Definition 2.1 *A compression body C is a 3-manifold obtained from an orientable connected closed surface Σ by attaching 2-handles to $\Sigma \times \{1\} \subset \Sigma \times [0, 1]$ and 3-balls to 2-sphere boundaries thereby created. We write*

$$\partial_+ C = \Sigma \times \{0\}, \qquad \partial_- C = \partial C - \partial_+ C.$$

When $C = \Sigma \times [0, 1]$, we say that C is a trivial compression body. When $\partial_- C = \emptyset$, we say that C is a handlebody.

Remark 2.1 If F is an essential annulus properly embeded in a compression body C, then this annulus must have one boundary component in $\partial_+ C$ as the other boundary component in $\partial_- C$. Furthermore, if F_1 and F_2 are two essential annuli such that $F_1 \bigcap \partial_+ C$ is isotopic to $F_2 \bigcap \partial_+ C$ in $\partial_+ C$, then F_1 is isotopic to F_2 in C.

Essential annuli play an important role in the following definition.

Definition 2.2 *For a compression body C, the annulus and disk complex $\mathcal{AD}(C)$ is defined as follows: A vertex of $\mathcal{AD}(C)$ is an isotopy class of an essential curve on $\partial_+ C$ which bounds an essential disk in C or cobounds an essential annulus in C with another curve in $\partial_- C$. $k+1$ vertices determine an k-simplex if and only if they can be represented by pairwise disjoint curves.*

Remark 2.2 If C is a trivial compression body, then $\mathcal{AD}(C)$ is nothing but the curve complex $\mathcal{C}(\partial_+ C)$. If C is a handlebody, then $\mathcal{AD}(C)$ is the disk complex $\Delta(C, \partial_+ C)$.

Then we define another complex associated to a compression body C without concerning $\partial_+ C$.

Definition 2.3 *For a compression body C, the complex $\widetilde{\mathcal{AD}}(C)$ is defined as follows: A vertex of $\widetilde{\mathcal{AD}}(C)$ is an isotopy class of an essential disk in C or an essential annulus in C. $k+1$ vertices $[F_0], \cdots, [F_k]$ determine an k-simplex if and only if we can isotopy $F_0, \cdots, F_k$ so that they are mutually disjoint.*

In fact, $\widetilde{\mathcal{AD}}(C)$ and $\mathcal{AD}(C)$ are isomophic.