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The Isomorphism Theorem of Regular Bisimple ω^2-semigroups

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Abstract: In this paper we study the isomorphisms of two regular bisimple ω^2-semigroups and obtain a criterion for two such semigroups to be isomorphic.

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1 Preliminaries

The structure of a bisimple ω-semigroup has been related by Reilly[1]. Earlier investigations in [2] we studied regular bisimple ω^2-semigroups, characterizing them as the generalized Bruck-Reilly extensions of groups. The results of [2] have generalized the results of regular bisimple ω-semigroups to regular bisimple ω^2-semigroups. There is often room for argument as to what constitutes a satisfactory structure theory in the semigroup context. Two questions have traditionally been asked. "Does there exist an associated isomorphism theorem?" and "Does the theory enable one to give an explicit description of the congruences?" The first question is inescapable, and applies in every algebraic context, but for an alleged structure theorem which does not have an associated isomorphism theorem is of little use. Reilly[1] has published these results for isomorphisms from a bisimple ω-semigroup to a bisimple ω-semigroup. In this paper, we give the necessary and sufficient conditions for two regular bisimple ω^2-semigroups to be isomorphic. We complete this section with a summary of notions of regular bisimple ω^2-semigroups, the details of which can be found in [1–4].

For any semigroup S, we denote by E_S the set of idempotents of S. Let S be a semigroup such that E_S is non-empty. We define a partial ordering $\geq$ on E_S by the rule that $e \geq f$

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if and only if $ef = f = fe$. To avoid ambiguity we sometimes denote a relation $\mathcal{K}$ on S by $\mathcal{K}(S)$. Let $\mathbf{N}^0$ denote the set of all non-negative integers and $\mathbf{N}$ denote the set of all positive integers. We define a partial order on $\mathbf{N}^0 \times \mathbf{N}^0$ in the following manner: if $(m, n), (p, q) \in \mathbf{N}^0 \times \mathbf{N}^0$, $(m, n) \le (p, q)$ if and only if $m > p$ or $m = p$ and $n \ge q$. The set $\mathbf{N}^0 \times \mathbf{N}^0$ with the above partial order is called an ω^2-chain, and is denoted by C_{ω^2}. Any partially ordered set order isomorphic to C_{ω^2} is also called an ω^2-chain. We say that a semigroup S is an ω^2-semigroup if and only if E_S is order isomorphic to C_{ω^2}. Thus, if S is an ω^2-semigroup, then we can write

$$E_S = \{e_{m,n} : m, n \in \mathbf{N}^0\},$$

where $e_{m,n} \le e_{p,q}$ if and only if $(m, n) \le (p, q)$.

Wang and Shang[2] has introduced the generalized Bruck-Reilly extension. Consider a monoid T with H_e as the $\mathcal{H}$-class which contains the identity e of T. Let β, γ be two homomorphisms from T into H_e. Let u be an element in H_e and τ_u be the inner automorphism of H_e defined by $x \to uxu^{-1}$ such that

$$\gamma\tau_u = \beta\gamma. \tag{1.1}$$

We can make $S = \mathbf{N}^0 \times \mathbf{N}^0 \times T \times \mathbf{N}^0 \times \mathbf{N}^0$ into a semigroup by defining

$$\begin{aligned}&(m, n, a, q, p)(m', n', a', q', p')\\ &= \begin{cases} (m,\, n - q + \max\{q, n'\},\, a\beta^{\max\{q,\, n'\}-q}a'\beta^{\max\{q,\, n'\}-n'}, & \\ \qquad\qquad\qquad q' - n' + \max\{q, n'\}, p') & \text{if } p = m';\\ (m,\, n,\, a(u^{-n'}a'\gamma u^{q'})\gamma^{p-m'-1}\beta^q,\, q,\, p' - m' + p) & \text{if } p > m';\\ (m - p + m',\, n',\, (u^{-n}a\gamma u^q)\gamma^{m'-p-1}\beta^{n'}a',\, q',\, p') & \text{if } p < m',\end{cases}\end{aligned}$$

where β^0, γ^0 are interpreted as the identity map of T and u^0 is interpreted as the identity e of T. Then S is a semigroup with identity $(0, 0, e, 0, 0)$. The semigroup $S = \mathbf{N}^0 \times \mathbf{N}^0 \times T \times \mathbf{N}^0 \times \mathbf{N}^0$ constructed above is called the generalized Bruck-Reilly extension of T determined by β, γ, u and is denoted by $S = \mathrm{GBR}(T; \beta, \gamma; u)$. Let $(m, n, a, q, p) \in S$. Then (m, n, a, q, p) is an idempotent if and only if $m = p$, $n = q$ and a is an idempotent.

Lemma 1.1[2] *Let $S = \mathrm{GBR}(T; \beta, \gamma; u)$ be a generalized Bruck-Reilly extension of a monoid T determined by β, γ, u. Suppose that (m, n, a, q, p) and (m', n', a', q', p') are elements in S. Then*

(i) $(m, n, a, q, p)\mathcal{L}(S)(m', n', a', q', p')$ *if and only if* $q = q'$, $p = p'$ *and* $a\mathcal{L}(T)a'$;

(ii) $(m, n, a, q, p)\mathcal{R}(S)(m', n', a', q', p')$ *if and only if* $m = m'$, $n = n'$ *and* $a\mathcal{R}(T)a'$.

Lemma 1.2[2] *Let $S = GBR(T; \beta, \gamma; u)$. Then an element (m, n, a, q, p) in S has an inverse $(x, y, b, z, w) \in S$ if and only if b is the inverse of a in T, $x = p$, $y = q$, $z = n$ and $w = m$.*

Lemma 1.3[2] *Let G be a group, β and γ be two endomorphisms of G, and u be an element of G. Let $S = GBR(G; \beta, \gamma; u)$ be the generalized Bruck-Reilly extension of G determined by β, γ and u. Then S is a regular bisimple ω^2-semigroup. Conversely, every regular bisimple ω^2-semigroup is isomorphic to some $GBR(G; \beta, \gamma; u)$.*