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A Note on Donaldson's "Tamed to Compatible" Question

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Abstract: Recently, Tedi Draghici and Weiyi Zhang studied Donaldson's "tamed to compatible" question (Draghici T, Zhang W. A note on exact forms on almost complex manifolds. arXiv: 1111. 7287v1 [math. SG]. Submitted on 30 Nov. 2011). That is, for a compact almost complex 4-manifold whose almost complex structure is tamed by a symplectic form, is there a symplectic form compatible with this almost complex structure? They got several equivalent forms of this problem by studying the space of exact forms on such a manifold. With these equivalent forms, they proved a result which can be thought as a further partial answer to Donaldson's question in dimension 4. In this note, we give another simpler proof of their result.
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1 Introduction

Donaldson[1] asked the following question:

Question 1.1 *For a compact almost complex* 4*-manifold* (M^4, J)*, if* J *is tamed by a symplectic form, is there a symplectic form compatible with* J*?*

An almost complex structure J on a manifold M^{2n} is tamed by a symplectic form ω, if ω is J-positive, i.e.,

$$\omega(X, JX) > 0, \qquad X \in TM, \ X \neq 0.$$

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An almost complex structure J is said to be compatible with ω, if ω is J-positive and J-invariant, i.e.,

$$\omega(JX, JY) = \omega(X, Y), \qquad X, Y \in TM.$$

Taubes[2] showed that this question has a positive answer on all compact almost complex 4-manifolds (M^4, J) with $b^+ = 1$ for generic almost complex structures. This problem is related to an almost-Kähler analogue of Yau's theorem (see [3]).

Draghici and Zhang[4] obtained the following result, which can be thought as a further partial answer to the Donaldson's question in dimension 4.

Theorem 1.1[4] *Let (M^4, J) be a compact almost complex manifold. The following statements are equivalent*:

(i) *J admits a compatible symplectic form*;

(ii) *For any J-anti-invariant form α, there exists a J-tamed symplectic form whose J-anti-invariant part is α.*

Draghici and Zhang[4] proposed several equivalent statements of Donaldson's question. If we use some notations as introduced in [4] (also see the next section for explanation), then Question 1.1 can be rewritten as

Question 1.2 *If $\mathcal{S}_J^t \neq \emptyset$, is $\mathcal{S}_J^c \neq \emptyset$ as well?*

We denote by $\mathcal{S}_J^c \neq \emptyset$ and $\mathcal{S}_J^t \neq \emptyset$ the set of symplectic forms, which are J-compatible and J-tamed, respectively.

Question 1.1 has the following two equivalent forms:

Question 1.3 *Is it true that either $\mathrm{d}\Omega^{J,-} \cap \mathrm{d}\Omega^{J,\oplus} = \emptyset$ or $\mathrm{d}\Omega^{J,\oplus} = \mathrm{d}\Omega^{J,+}$?*

Question 1.4 *If $\alpha \in \Omega^{J,-}$ satisfies $\mathrm{d}\alpha \in \mathrm{d}\Omega^{J,\oplus}$, is it true that $\mathrm{d}(-\alpha) \in \mathrm{d}\Omega^{J,\oplus}$ as well?*

For a detailed proof of equivalent statements, we refer to [4].

Let (M^4, J) be a compact almost complex manifold. $\wedge^2(M)$ is the vector bundle of (real) 2-forms on M. $\Omega^2(M)$ denotes the space of real C^∞ 2-forms, i.e., the C^∞ sections of the bundle $\wedge^2(M)$.

J acts on $\Omega^2(M)$ as an involution via

$$J:\ \Omega^2(M) \longrightarrow \Omega^2(M),$$
$$\alpha \longmapsto \alpha^J,$$

where $\alpha^J(\cdot\,, \cdot\,) = \alpha(J\,\cdot\,, J\,\cdot\,)$. Thus, we can define J-invariant forms and J-anti-invariant forms by

$$\Omega^{J,+}(M) := \{\alpha \in \Omega^2(M) \mid \alpha^J = \alpha\},$$
$$\Omega^{J,-}(M) := \{\alpha \in \Omega^2(M) \mid \alpha^J = -\alpha\}.$$

It is easy to see that $\Omega^{J,+}(M)$ and $\Omega^{J,-}(M)$ are vector spaces and we have

$$\Omega^2(M) = \Omega^{J,+}(M) \oplus \Omega^{J,-}(M).$$