Communications in Mathematical Research
30(2)(2014), 183–192

Intrinsic Knotting of Almost Complete Partite Graphs

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Communicated by Lei Feng-chun

Abstract: Let G be a complete p-partite graph with 2 edges removed, $p \geq 7$, which is intrinsically knotted. Let J represent any graph obtained from G by a finite sequence of Δ-Y exchanges and/or vertex expansions. In the present paper, we show that the removal of any vertex of J and all edges incident to that vertex produces an intrinsically linked graph. This result offers more intrinsically knotted graphs which hold for the conjecture presented in Adams' book (Adams C. The Knot Book. New York: W. H. Freeman and Company, 1994), that is, the removal of any vertex from an intrinsically knotted graph yields an intrinsically linked graph.

Key words: intrinsically knotted graph, Δ-Y exchange, vertex-expansion
2010 MR subject classification: 57M25, 57M27
Document code: A
Article ID: 1674-5647(2014)02-0183-10
DOI: 10.13447/j.1674-5647.2014.02.09

1 Introduction

A graph is intrinsically knotted (IK) if every embedding of it in $\mathbf{R}^3$ contains a non-trivially knotted cycle. Similarly, a graph is intrinsically linked (IL) if every embedding of it in $\mathbf{R}^3$ contains a pair of non-trivially linked cycles.

Robertson *et al.*[1] showed that the intrinsic linking is determined by the seven Petersen graphs. A graph is intrinsically linked if and only if it contains one of the seven as a minor. Since knotless embedding is preserved under edge contraction (see [2]), Robertson and Seymour[3] demonstrated that a similar, finite list of graphs exists for the intrinsic knotting property. However, determining such a set of graphs remains difficult.

It is known that K_7 and $K_{3,3,1,1}$ along with any graph obtained from these two by Δ-Y

Received date: Jan. 6, 2014.
Foundation item: The Scientific Research Common Program (KM201410011006) of Beijing Municipal Commission of Education, the Research Foundation (QNJJ2012-26) for Youth Scholars of Beijing Technology and Business University, the NSF (1122013, 1132002) of Beijing and the Importation and Development of High-caliber Talents Project (CIT&TCD201304029) of Beijing Municipal Institutions.
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exchanges are minor minimal with respect to intrinsic knotting (see [4–7]). Foisy[8] added a new minor minimal graph to the list, moreover, provided a counterexample to the "unsolved question" posed in Adams' book (see [9]): "Whether removing a vertex and all edges incident to that vertex from an intrinsically knotted graph must yield an intrinsically linked graph". For more counterexamples, please refer to [10].

While Adams' conjecture is not true in general, it does seem to hold for a large class of graphs (see [9]). A graph is k-deficient if it is a complete or complete partite graph with k ($k = 0, 1, 2, \cdots$) edges removed. Campbell *et al.*[11] have classified all 0-, 1-, 2-deficient graphs with respect to intrinsic linking and intrinsic knotting, all of which are shown to be hold for Adams' conjecture. It is shown in [12] that any graph obtained from a 0- or 1-deficient IK graph by a finite sequence of Δ-Y exchanges and/or vertex expansions satisfies Adams' conjecture.

In this paper, we extend the results in [12] to any graph obtained from complete p-partite graphs ($p \geq 7$) with 2 edges removed by a finite sequence of Δ-Y exchanges and/or vertex expansions, and present a larger array of graphs which hold for Adams' conjecture. In Section 2, we include some necessary definitions and preliminaries. In Section 3, we state the main theorems and give their proofs.

2 Preliminaries

Throughout the paper we take an embedded graph to mean a graph embedded in 3-space, where all the embeddings are tame. In this section, we present some useful lemmas and notations which are used throughout the paper.

K_n denotes the complete graph on n vertices, and $K_{n_1,n_2,\cdots,n_p}$ denotes the complete p-partite graph, $p \geq 2$, with n_i vertices in the i-th part. $K_{n_1,n_2,\cdots,n_p} - k$, $p \geq 1$, $k \geq 0$, denotes the graph obtained by removing k edges from $K_{n_1,n_2,\cdots,n_p}$. Note here that deleting any Δ from a graph mentioned throughout the paper means deleting the edges of Δ from the graph. And for the special case that graph G contains no triangle, we assume $G - \Delta \equiv G$.

Recall that a vertex expansion of a vertex v in a graph G is achieved by replacing v with two vertices v' and v'', adding the edge (v', v''), and connecting a subset of the edges that are incident with v to v', and connecting the remaining edges that are incident with v to v''. The reverse of this operation is edge contraction.

Let G be a graph which contains a triangle Δ. Remove the three edges of Δ from G. Add three new edges, connecting the three vertices of Δ to a new vertex. The resulting graph G' is said to be obtained from G by a Δ-Y move.

A minor of a graph G is the graph resulting from a finite number of vertex deletions, edge deletions and edge contractions on G. If we know that a graph G contains a linked (knotted) minor, then G must also be linked (knotted). Conversely, if we can realize G as a minor of an unlinked (unknotted) graph, then G must also be unlinked (unknotted).

Lemma 2.1[11] *$K_{n_1+n_2,n_3,\cdots,n_p}$ is a minor of $K_{n_1,n_2,n_3,\cdots,n_p}$, where $p \geq 3$. Similarly,*