COMMUNICATIONS IN MATHEMATICAL RESEARCH
30(3)(2014), 193–200

Bounded 3-manifolds with Distance n Heegaard Splittings

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Communicated by Lei Feng-chun

Abstract: We prove that for any integer $n \geq 2$ and $g \geq 2$, there are bounded 3-manifolds admitting distance n, genus g Heegaard splittings with any given boundaries.

Key words: attaching compression body, Heegaard distance, subsurface projection
2010 MR subject classification: 57M27
Document code: A
Article ID: 1674-5647(2014)03-0193-08
DOI: 10.13447/j.1674-5647.2014.03.01

1 Introduction

Let S be a compact surface with $\chi(S) \leq -2$ but not a 4-punctured sphere. Harvey[1] defined the curve complex $\mathcal{C}(S)$ as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S, and $k+1$ distinct vertices $x_0, x_1, \cdots, x_k$ determine a k-simplex of $\mathcal{C}(S)$ if and only if they are represented by pairwise disjoint simple closed curves. For two vertices x and y of $\mathcal{C}(S)$, the distance of x and y, denoted by $d_{\mathcal{C}(S)}(x, y)$, is defined to be the minimal number of 1-simplexes in a simplicial path joining x to y. In other words, $d_{\mathcal{C}(S)}(x, y)$ is the smallest integer $n \geq 0$ such that there is a sequence of vertices $x_0 = x$, $\cdots$, $x_n = y$ such that x_{i-1} and x_i are represented by two disjoint essential simple closed curves on S for each $1 \leq i \leq n$. For two sets of vertices in $\mathcal{C}(S)$, say X and Y, $d_{\mathcal{C}(S)}(X, Y)$ is defined to be $\min\{d_{\mathcal{C}(S)}(x, y) \mid x \in X, y \in Y\}$. Now let S be a torus or a once-punctured torus. In this case, Masur and Minsky[2] defined $\mathcal{C}(S)$ as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S, and $k+1$ distinct vertices x_0, $x_1, \cdots, x_k$ determine a k-simplex of $\mathcal{C}(S)$ if and only if x_i and x_j are represented by two simple closed curves c_i and c_j on S such that c_i intersects c_j in just one point for each

Received date: May 9, 2013.
Foundation item: The NSF (11271058 and 11371076) of China.
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$0 \le i \ne j \le k$.

Let M be a compact orientable 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W such that $S = \partial_+ V = \partial_+ W$, then we say that M has a Heegaard splitting, denoted by $M = V \cup_S W$, where $\partial_+ V$ (resp. $\partial_+ W$) means the positive boundary of V (resp. W). We denote by $\mathcal{D}(V)$ (resp. $\mathcal{D}(W)$) the set of vertices in $\mathcal{C}(S)$ such that each element of $\mathcal{D}(V)$ (resp. $\mathcal{D}(W)$) is represented by the boundary of an essential disk in V (resp. W). The distance of the Heegaard splitting $V \cup_S W$, denoted by $d(S)$, is defined to be $d_{\mathcal{C}(S)}(\mathcal{D}(V), \mathcal{D}(W))$ (see [3]).

Hempel[3] showed that for any integers $g \ge 2$ and $n \ge 2$, there is a 3-manifold admitting a distance at least n Heegaard splitting of genus g. Similar results are obtained in different ways by [4–5]. Minsky, Moriah and Schleimer[6] proved the same result for knot complements, and Li[7] constructed the non-Haken manifolds admitting high distance Heegaard splittings. In general, generic Heegaard splittings have Heegaard distances at least n for any $n \ge 2$ (see [8–10]). By studying Dehn filling, Ma *et al.*[11] proved that distances of genus 2 Heegaard splittings cover all non-negative integers except 1. Recently, Ido *et al.*[12] proved that, for any $n > 1$ and $g > 1$, there is a compact 3-manifold with two boundary components which admits a distance n Heegaard splitting of genus g. Johnson[13] proved that there always exist closed 3-manifolds admitting a distance $n \ge 5$, genus g Heegaard splitting. Qiu *et al.*[14] proved that there is closed 3-manifold admitting any given distance, genus Heegaard splitting.

The main result of this paper is the following theorem:

Theorem 1.1 *Let n be a positive integer, $\{F_1, \cdots, F_n\}$ be a collection of closed orientable surfaces, $I \subset \{1, 2, \cdots, n\}$ and $J = \{1, \cdots, n\} \backslash I$ be two subsets of $\{1, \cdots, n\}$. Then, for any integers*

$$g \ge \max\Big\{\sum_{i\in I} g(F_i), \ \sum_{j \in J} g(F_j)\Big\}, \qquad m \ge 2,$$

there is a compact 3-manifold M admitting a distance m Heegaard splitting of genus g, say $M = V \cup_S W$, such that

$$F_i \subset \partial_- V, \quad i \in I, \qquad F_j \subset \partial_- W, \quad j \in J.$$

We introduce some results of curve complex in Section 2 and prove the main theorem in Section 3.

2 Some Results of Curve Complex

Let S be a compact surface of genus at least 1, and $\mathcal{C}(S)$ be the curve complex of S. We say that a simple closed curve c in S is essential if c bounds no disk in S and is not parallel to ∂S. Hence each vertex of $\mathcal{C}(S)$ is represented by the isotopy class of an essential simple closed curve in S. For simplicity, we do not distinguish the essential simple closed curve c and its isotopy class c without any further notation. The following lemma is well known (see [2], [15–16]).