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λ-central BMO Estimates for Higher Order Commutators of Hardy Operators

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Abstract: In this paper, the λ-central BMO estimates for higher order commutators of Hardy operators on central Morrey space $L^{q,\lambda}(\mathbf{R}^n)$ are established. In the meanwhile, the corresponding corollary for central BMO estimates is also obtained.

Key words: λ-central BMO space, commutator, Hardy operator, central Morrey space

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1 Introduction and Main Result

Let f be a locally integrable function on $\mathbf{R}^n$. The n-dimensional Hardy operators are defined by

$$\mathscr{H}(f)(x) \triangleq \frac{1}{|x|^n}\int_{|t|<|x|} f(t)\mathrm{d}t,$$

$$\mathscr{H}^*(f)(x) \triangleq \int_{|t|\geq|x|} \frac{f(t)}{|t|^n}\mathrm{d}t, \qquad x \in \mathbf{R}^n \setminus \{0\}.$$

In 1995, Christ and Grafakos[1] obtained results for the boundedness of $\mathscr{H}$ on $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ spaces. They also found the exact operator norms of $\mathscr{H}$ on $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ spaces.

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It is easy to see that $\mathscr{H}$ and $\mathscr{H}^*$ satisfy

$$\int_{\mathbf{R}^n} g(x)\mathscr{H}(f)(x)\mathrm{d}x = \int_{\mathbf{R}^n} f(x)\mathscr{H}^*(g)(x)\mathrm{d}x.$$

And we have $|\mathscr{H}(f)(x)| \le C_n Mf(x)$, where M is the Hardy-Littlewood maximal operator which is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(t)|\mathrm{d}t,$$

where the supremum is taken over all balls containing x.

In 2007, Fu *et al.*[2] gave the central BMO estimates for commutators, $\mathscr{H}_b(f)$ and $\mathscr{H}_b^*(f)$, of n-dimensional Hardy operators, where b is a locally integrable function on $\mathbf{R}^n$, $\mathscr{H}_b(f)$ and $\mathscr{H}_b^*(f)$ are defined as follows:

$$\mathscr{H}_b(f) \triangleq b\mathscr{H}(f) - \mathscr{H}(bf), \qquad \mathscr{H}_b^*(f) \triangleq b\mathscr{H}^*(f) - \mathscr{H}^*(bf).$$

In 2000, Alvarez *et al.*[3] studied the relationship between central BMO spaces and Morrey spaces. Furthermore, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces, respectively.

Definition 1.1[3–4] (λ-central BMO space) *Let $1 < q < \infty$ and $-\frac{1}{q} < \lambda < \frac{1}{n}$. A function $f \in L^q_{\mathrm{loc}}(\mathbf{R}^n)$ is said to belong to the λ-central bounded mean oscillation space $\mathrm{CBMO}^{q,\lambda}(\mathbf{R}^n)$, if*

$$\|f\|_{\mathrm{CBMO}^{q,\lambda}(\mathbf{R}^n)} = \sup_{r>0} \left\{ \frac{1}{|B(0,\, r)|^{1+\lambda q}} \int_{B(0,\, r)} |f(x) - f_B|^q \mathrm{d}x \right\}^{\frac{1}{q}} < \infty, \tag{1.1}$$

where

$$B(0,\, r) = \{x \in \mathbf{R}^n : |x| < r\}, \qquad f_B = |B(0,\, r)|^{-1} \int_{B(0,\, r)} f(x)\mathrm{d}x,$$

and $|B(0,\, r)|$ is the measure of $B(0,\, r)$.

Remark 1.1 If two functions which differ by a constant are regarded as a function in the space $\mathrm{CBMO}^{q,\lambda}(\mathbf{R}^n)$, then $\mathrm{CBMO}^{q,\lambda}(\mathbf{R}^n)$ becomes a Banach space. Apparently, (1.1) is equivalent to the condition (see [3–4])

$$\sup_{r>0} \inf_{c \in \mathbf{C}} \left\{ \frac{1}{|B(0,\, r)|^{1+\lambda q}} \int_{B(0,\, r)} |f(x) - c|^q \mathrm{d}x \right\}^{\frac{1}{q}} < \infty.$$

Definition 1.2[3–4] (Central Morrey spaces) *Let $1 < q < \infty$ and $-\frac{1}{q} < \lambda < 0$. The central Morrey space $L^{q,\lambda}(\mathbf{R}^n)$ is defined by*

$$\|f\|_{L^{q,\lambda}(\mathbf{R}^n)} = \sup_{r>0} \left\{ \frac{1}{|B(0,\, r)|^{1+\lambda q}} \int_{B(0,\, r)} |f(x)|^q \mathrm{d}x \right\}^{\frac{1}{q}} < \infty, \tag{1.2}$$

where

$$B(0,\, r) = \{x \in \mathbf{R}^n : |x| < r\},$$

and $|B(0,\, r)|$ is the measure of $B(0,\, r)$.

Remark 1.2 It follows from (1.1) and (1.2) that $L^{q,\lambda}(\mathbf{R}^n)$ is a Banach space continuously included in $\mathrm{CBMO}^{q,\lambda}(\mathbf{R}^n)$.