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The Dependence Problem for a Class of Polynomial Maps in Dimension Four

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Abstract: Let h be a polynomial in four variables with the singular Hessian $\mathcal{H}h$ and the gradient ∇h and R be a nonzero relation of ∇h. Set $H = \nabla R(\nabla h)$. We prove that the components of H are linearly dependent when $\mathrm{rk}\mathcal{H}h \leq 2$ and give a necessary and sufficient condition for the components of H to be linearly dependent when $\mathrm{rk}\mathcal{H}h = 3$.

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1 Introduction

Throughout this paper k denotes a field of characteristic 0, and $k[X]:=k[x_1, x_2, \cdots, x_n]$ denotes the polynomial ring in the variables $x_1, x_2, \cdots, x_n$ over k.

The linear dependence problem asks whether the components of a polynomial map $H : k^n \to k^n$ are linearly dependent over k if the Jacobian matrix $\mathcal{J}H$ is nilpotent. Partial positive answers to the problem are obtained in [1–3]. By studying quasi-translation De Bondt[4–5] solved the problem negatively for all $n \geq 5$ in the homogeneous case and for all $n \geq 4$ in the non-homogeneous case. A polynomial map $X + H$ is called a quasi-translation if its inverse is $X - H$. De Bondt[5] furthermore gave examples of quasi-translations with the components of H linearly independent for $n \geq 6$ in the homogeneous case and for $n \geq 4$ in the non-homogeneous case, and he also proved that no such examples exist when $n \leq 4$ for the homogeneous case and when $n \leq 3$ for the non-homogeneous case.

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For a polynomial $h \in k[X]$, denote by $\mathcal{H}h$ its Hessian matrix and by ∇h its gradient. If $R(Y) \in k[Y]$ is a relation of ∇h, that is, $R(\nabla h) = 0$, we set

$$H := \nabla R(\nabla h) = \left(\frac{\partial R}{\partial x_1}(\nabla h),\ \frac{\partial R}{\partial x_2}(\nabla h),\ \cdots,\ \frac{\partial R}{\partial x_n}(\nabla h) \right).$$

De Bondt[5] proved that $X + H$ is a quasi-translation, called quasi-translation corresponding to h, and he asked whether the components of H are linearly dependent.

As mentioned above, for $n \leq 3$ the answer to the problem of De Bondt is affirmative and it is also affirmative in the case $n = 4$ and H is homogenous. In this paper, we study the problem for $n = 4$. We prove that the components of H are linearly dependent if the rank $\mathrm{rk}\mathcal{H}h \leq 2$. For the case $\mathrm{rk}\mathcal{H}h = 3$ and $H \neq 0$, we prove that the components of H are linearly dependent if and only if the components of ∇g are linearly dependent, where g is a generator of the relation ideal of ∇h. Finally, we give an algorithm to decide whether the components of H are linearly dependent.

2 Main Results

For $g, h \in k[X]$, we say that they are linearly equivalent, if there exists a $T \in Gl_n(k)$ such that $g = h(TX)$. In this case, $\nabla g = T^t \nabla h(TX)$, $\mathcal{H}g = T^t \mathcal{H}h(TX)T$, and $\mathrm{rk}\mathcal{H}g = \mathrm{rk}\mathcal{H}h$, where T^t denotes the transpose of T.

Lemma 2.1 *Suppose that $g, h \in k[X]$ are linearly equivalent. Then there is a nonzero relation R of ∇h such that the components of $\nabla R(\nabla h)$ are linearly dependent if and only if there is a nonzero relation S of ∇g such that the components of $\nabla S(\nabla g)$ are linearly dependent.*

Proof. It suffices to prove the assertion for one direction by the definition of linear equivalence. Let $g = h(TX)$ for some $T \in Gl_n(k)$ and $R \in k[Y] := k[y_1, \cdots, y_n]$ be a nonzero relation of ∇h such that the components of $H = \nabla R(\nabla h)$ are linearly dependent. Suppose $0 \neq \lambda \in k^n$ such that $\lambda H = 0$. Take $S(Y) = R((T^t)^{-1}Y)$. Then

$$S(\nabla g) = S(T^t \nabla h(TX)) = R((T^t)^{-1}Y) \mid_{Y = T^t \nabla h(TX)} = R(\nabla h(TX)) = 0.$$

Let $G = \nabla S(\nabla g)$. Note that

$$\begin{aligned} \nabla S(\nabla g) &= T^{-1} \nabla R\big((T^t)^{-1} Y\big) \mid_{Y = \nabla g} \\ &= T^{-1} \nabla R\big((T^t)^{-1} \nabla g\big) \\ &= T^{-1} \nabla R\big((T^t)^{-1} \big(T^t \nabla h(TX)\big)\big) \\ &= T^{-1} \nabla R(\nabla h(TX)). \end{aligned}$$

Let $\beta = \lambda T$. Then $\beta \neq 0$ and

$$\beta G = \beta T^{-1} \nabla R(\nabla h(TX)) = \lambda T T^{-1} \nabla R(\nabla h(TX)) = \lambda H(TX) = 0,$$

as desired.

For $h \in k[X]$ and a relation R of ∇h, let $H = \nabla R(\nabla h)$. Taking Jacobian matrix on both sides, we have $\mathcal{J}H = \mathcal{J}(\nabla R) \mid_{X = \nabla h} \mathcal{H}(h)$. Hence $\mathrm{rk}\mathcal{J}H \leq \mathrm{rk}\mathcal{H}h$.