Communications
in
Mathematical
Research
30(4)(2014), 301–306

Jordan Left Derivations of Generalized Matrix Algebras

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Communicated by Du Xian-kun

Abstract: In this paper, it is proved that under certain conditions, each Jordan left derivation on a generalized matrix algebra is zero and each generalized Jordan left derivation is a generalized left derivation.

Key words: generalized Jordan left derivation, Jordan left derivation, generalized matrix algebra, generalized left derivation

2010 MR subject classification: 15A78, 16W25, 47L35

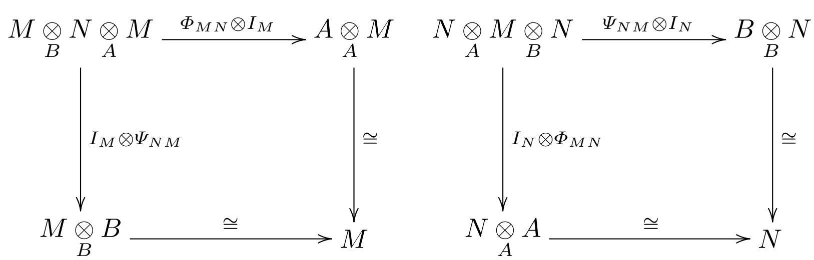
Document code: A

Article ID: 1674-5647(2014)04-0301-06

DOI: 10.13447/j.1674-5647.2014.04.03

1 Introduction

Let us begin with the definition of generalized matrix algebras given by a Morita context. Let $\mathcal{R}$ be a commutative ring with identity. A Morita context consists of two $\mathcal{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \underset{B}{\otimes} N \longrightarrow A$ and $\Psi_{NM} : N \underset{A}{\otimes} M \longrightarrow B$ satisfying the following commutative diagrams:



Received date: Jan. 30, 2013.
Foundation item: Fundamental Research Funds (N110423007) for the Central Universities.
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Let us write this Morita context as $(A,\ B,\ {}_AM_B,\ {}_BN_A,\ \Phi_{MN},\ \Psi_{NM})$. If $(A,\ B,\ {}_AM_B,\ {}_BN_A,$ $\Phi_{MN},\ \Psi_{NM})$ is a Morita context, then the set

$$\left[\begin{array}{cc} A & M \\ N & B \end{array}\right] = \left\{ \left[\begin{array}{cc} a & m \\ n & b \end{array}\right] \middle| \ a \in A,\ m \in M,\ n \in N,\ b \in B \right\}$$

forms an $\mathcal{R}$-algebra under matrix-like addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. Such an $\mathcal{R}$-algebra is called a generalized matrix algebra and is usually denoted by $\boldsymbol{\mathcal{G}} = \left[\begin{array}{cc} A & M \\ N & B \end{array}\right]$. Li and Wei[1–3], Li and Xiao[4], Xiao and Wei[5] jointly studied linear mappings of generalized matrix algebras, such as derivations, Lie derivations, commuting mappings and semi-centralizing mappings.

Let $\mathcal{A}$ be a unital algebra over $\mathcal{R}$, and $\mathcal{M}$ be a left $\mathcal{A}$-module. Recall that an $\mathcal{R}$-linear map f from $\mathcal{A}$ into $\mathcal{M}$ is called a left derivation if $f(ab) = af(b) + bf(a)$ and called a generalized left derivation if there exists another $\mathcal{R}$-linear map d such that $f(ab) = af(b) + bd(a)$. Further, an $\mathcal{R}$-linear map J from $\mathcal{A}$ into $\mathcal{M}$ is called a Jordan left derivation if $J(a^2) = 2aJ(a)$ for all $a \in \mathcal{A}$, and called a generalized Jordan left derivation if $J(a^2) = aJ(a) + ad(a)$, where d is a Jordan left derivation. Let $\mathcal{M}$ be a $\mathcal{A}$-$\mathcal{A}$-bimodule. An $\mathcal{R}$-linear map f from $\mathcal{A}$ into $\mathcal{M}$ is called a derivation if $f(ab) = f(a)b + af(b)$ for all $a, b \in \mathcal{A}$, a generalized derivation if $f(ab) = f(a)b + ad(b)$, where d is a derivation from $\mathcal{A}$ to $\mathcal{M}$ and a Jordan derivation if $J(a^2) = J(a)a + aJ(a)$ for all $a \in \mathcal{A}$.

Note that the concept of Jordan left derivation were introduced by Brešar and Vukman[6]. Han and Wei[7] studied generalized Jordan left derivations of semiprime algebras. They proved that any generalized Jordan left derivation on a semiprime algebra of characteristic not 2 is a generalized left derivation and is also a generalized derivation. The aim of this note is to study Jordan left derivations and generalized Jordan left derivations of generalized matrix algebras. We prove that under certain conditions, each Jordan left derivation on $\boldsymbol{\mathcal{G}}$ is zero. As a by-product, we get that under certain conditions, each generalized Jordan left derivation of $\boldsymbol{\mathcal{G}}$ is a generalized left derivation.

2 Jordan Left Derivations of Generalized Matrix Algebras

From now on, we always assume that all algebras and modules considered in this note are 2-torsion free. Let us first describe the general form of Jordan left derivations on generalized matrix algebras.

Lemma 2.1 *An additive map Θ from $\boldsymbol{\mathcal{G}}$ into itself is a Jordan left derivation if and only if it is of the form*:

$$\Theta\left(\left[\begin{array}{cc} a & m \\ n & b \end{array}\right]\right) = \left[\begin{array}{cc} \delta_1(a) & \tau_1(a) \\ \nu_4(b) & \mu_4(b) \end{array}\right], \qquad \left[\begin{array}{cc} a & m \\ n & b \end{array}\right] \in \boldsymbol{\mathcal{G}},$$

where

$$\delta_1 : A \longrightarrow A, \quad \tau_1 : A \longrightarrow M, \quad \nu_4 : B \longrightarrow N, \quad \mu_4 : B \longrightarrow B$$

are all $\mathcal{R}$-linear maps satisfying the following conditions: