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A Note on Generalized Long Modules

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Abstract: Let ${}_H\mathcal{L}^B$ be the category of generalized Long modules, that is, H-modules and B-comodules over Hopf algebras B and H. We describe a new Turaev braided group category over generalized Long module ${}_H\mathcal{L}^B(\mathscr{S}(\pi))$ where the opposite group $\mathscr{S}(\pi)$ of the semidirect product of the opposite group $\pi^{\rm op}$ of a group π by π. As an application, we show that this is a Turaev braided group-category ${}_H\mathcal{L}^B$ for a quasi-triangular Turaev group-coalgebra H and a coquasitriangular Turaev group-algebra B.

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1 Introduction

For a group π, Turaev[1] introduced the notion of a braided π-monoidal category, here called Turaev braided π-category, and showed that such a category gives rise to a 3-dimensional homotopy quantum field theory. Kirillov[2] found that such Turaev braided π-categories also provide a suitable mathematical tool to describe the orbifold models which arise in the study of conformal field theories. Virelizier[3] used Turaev braided π-category to construct Hennings-type invariants of flat π-bundles over complements of links in the 3-sphere. We note that a Turaev braided π-category is a braided monoidal category when π is trivial.

Starting from the category of Yetter-Drinfeld modules, Panaite and Staic[4] constructed

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a Turaev braided category over certain group, generalizing the work of Staic[5]. Some recent related works can be found in [6–8].

The paper is organized as follows. In Section 2, we show that the braided monoidal structure on the category ${}_H\mathcal{L}^B$ depends on additional compatibility conditions (see Theorem 2.3). In Section 3, we find that ${}_H\mathcal{L}^B(\mathscr{S}(\pi))$ is a Turaev braided $\mathscr{S}(\pi)$-category if and only if there is a linear map $\mathscr{Q} : B \otimes B \longrightarrow H \otimes H$ satisfying some conditions (see Theorem 3.8). In Section 4, as an application, for a quasitriangular Turaev $\mathscr{S}(\pi)$-coalgebra H and a coquasitriangular Turaev $\mathscr{S}(\pi)$-algebra B, we show that ${}_H\mathcal{L}^B(\mathscr{S}(\pi))$ is a Turaev braided $\mathscr{S}(\pi)$-category (see Theorem 4.4).

2 Preliminaries

Throughout the paper, let k be a fixed field and $\otimes$ be over k. For a Hopf algebra H, we always denote by $\mathrm{Aut}(H)$ the group of Hopf automorphism of H. For the comultiplication $\varDelta$ in a coalgebra C, we use the Sweedler-Heyneman's notation (see [9]):

$$\varDelta(c) = c_1 \otimes c_2, \qquad c \in C.$$

We denote by Mod^B the category of the right B-comodules and for any $M \in \mathrm{Mod}^B$, we write

$$\rho(m) = m_{(0)} \otimes m_{(1)}, \qquad m \in M.$$

Similarly, we have the ${}_H\mathrm{Mod}$ of left H-modules.

Definition 2.1[1] *Let π be a group with the unit e. Recall from* [1] *that a Turaev π-category is a monoidal category $\mathcal{C}$ which consists of the following data*:

(1) *A family of subcategories $\{\mathcal{C}_\alpha\}_{\alpha\in\pi}$ such that $\mathcal{C}$ is a disjoint union of this family and such that $U \otimes V \in \mathcal{C}_{\alpha\beta}$, for any $\alpha, \beta \in \pi$, if the $U \in \mathcal{C}_\alpha$ and $V \in \mathcal{C}_\beta$. Here the subcategory $\mathcal{C}_\alpha$ is called the αth component of $\mathcal{C}$*;

(2) *A group homomorphism $\varphi : \pi \longrightarrow \mathrm{aut}(\mathcal{C})$, $\beta \mapsto \varphi_\beta$, the conjugation,* (*where* $\mathrm{aut}(\mathcal{C})$ *is the group of invertible strict tensor functors from $\mathcal{C}$ to itself*) *such that $\varphi_\beta(\mathcal{C}_\alpha) = \mathcal{C}_{\beta\alpha\beta^{-1}}$ for any $\alpha, \beta \in \pi$. Here the functors φ_β are called conjugation isomorphisms.*

We use the left index notation (*see* [8, 10]): *Given $\beta \in G$ and an object $V \in \mathcal{C}_\beta$, the functor φ_β will be denoted by ${}^V(\,\cdot\,)$ or ${}^\beta(\,\cdot\,)$. We use the notation ${}^{\overline{V}}(\,\cdot\,)$ for ${}^{\beta^{-1}}(\,\cdot\,)$. Then we have ${}^V\mathrm{id}_U = \mathrm{id}_{{}^VU}$ and ${}^V(g \circ f) = {}^Vg \circ {}^Vf$. We remark that since the conjugation $\varphi : \pi \longrightarrow \mathrm{aut}(\mathcal{C})$ is a group homomorphism, for any $V, W \in \mathcal{C}$, we have*

$${}^{V\otimes W}(\,\cdot\,) = {}^V({}^W(\,\cdot\,)), \qquad {}^1(\,\cdot\,) = {}^V({}^{\overline{V}}(\,\cdot\,)) = {}^{\overline{V}}({}^V(\,\cdot\,)) = \mathrm{id}_{\mathcal{C}},$$

and that since, for any $V \in \mathcal{C}$, the functor ${}^V(\,\cdot\,)$ is strict, we have ${}^V(f \otimes g) = {}^Vf \otimes {}^Vg$, for any morphism f and g in $\mathcal{C}$, and ${}^V1 = 1$. And we use $\mathcal{C}(U, V)$ to denote a set of morphisms (*or arrows*) *from U to V in $\mathcal{C}$.*

A Turaev braided π-category is a Turaev π-category $\mathcal{C}$ endowed with a braiding, i.e., with a family of isomorphisms

$$\tau = \{\tau_{U,V} \in \mathcal{C}(U \otimes V,\ ({}^UV) \otimes U)\}_{U,V\in\mathcal{C}}$$