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A Note on the Connectedness of the Invertible Group of a Nest Algebra

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Abstract: The connectedness of the invertibles question for arbitrary nest has been reduced to the case of the lower triangular operators with respect to a fixed orthonormal basis e_n for $n \geqslant 1$. For each $f \in H^\infty$, let T_f be the Toeplitz operator. In this paper we prove that T_f can be connected to the identity through a path in the invertible group of the lower triangular operators if f satisfies certain conditions.

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1 Introduction

A well-known problem in operator theory is whether the group of invertible operators in a nest algebra is connected in the norm topology. David Larson remembered people in system theory raising it in late 1970s. William Arveson promoted this problem for many years, in particular, discussed it at a meeting in Berkeley in 1981 in honour of Paul Halmos. Davidson also raised this problem as one of the ten important problems in that area in a talk at the 1987 GPOTS meeting in Kansas (see [1] and Chapter 25 in [2]). In spite of all this, there has been no significant progress until the emergence of the deep interpolation theorem of Orr[3]. In 1993, Davidson and Orr[4] achieved a breakthrough in solving the connectedness problem. By using the interpolation theorem of Orr, they showed that the invertibles are connected in each nest algebra of infinite multiplicity. This is the first significant progress about the connectedness problem. And soon after, in 1994, Davidson, Orr and Pitts[5] successfully extended these results by showing that the invertibles are connected in a nest algebra provided there is a finite bound on the number of consecutive finite rank atoms in

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the nest. Their work reduced the connectedness of the invertibles question for arbitrary nests to the case of the lower triangular operators with respect to a fixed orthonormal basis e_n for $n \geq 1$.

Let $\mathbb{T}$ be the unit circle in the complex plane with normalized Lesbegue measure. For $1 \leq p \leq \infty$, let H^p be the usual Hardy space of all functions in $L^p(\mathbb{T})$ which have analytic extensions to the open unit disk $\mathbb{D}$. Let $\mathcal{H} = H^2(\mathbb{T})$ and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$, and $W \in B(\mathcal{H})$ be the shift operator with $(Wf)(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta} f(\mathrm{e}^{\mathrm{i}\theta})$. In this paper we consider the nest $\mathbb{N} = \{\{0\}, W^n\mathcal{H} : n \in \mathbf{Z},\ n \geq 0\}$ of subspaces of $\mathcal{H}$ and its associated nest algebra

$$\mathrm{Alg}\mathbb{N} = \{T \in B(\mathcal{H}) :\ TW^n\mathcal{H} \subseteq W^n\mathcal{H}\}.$$

Is the group of invertible elements of Banach algebra AlgN connected in the norm topology? It is frequently conjectured that the answer to this question is “no”. The reason for conjecturing a negative answer is due to a strong analogy between nest algebras and analytic function theory. Pitts[6] proposed a function f which cannot be connected to the constant function 1 via a norm continuous path within the group of invertible elements of the Banach algebra H^∞, however the Toeplitz operator with symbol f can be connected to the identity via a norm continuous path of invertible elements in AlgN. Inspired by Pitts[6] we show that a certain class of invertible elements in AlgN can be connected to the identity in norm topology in this paper.

The main results are the theorems as follows.

Theorem 1.1 *If $T \in \mathrm{Alg}\mathbb{N}$ is invertible and $\sigma(T)$ consists of finite points, then T can be connected to the identity through a path in the invertible group of* AlgN.

Theorem 1.2 *Let h be an analytic function in the open unit disc $\mathbb{D}$ with the bounded real part and $\hat{h}(n)$ be the n-th Fourier coefficient of h. If there exists a natural number k such that $\hat{h}(kl) = 0$ for each $l \in \mathbf{Z}$, then the Toeplitz operator T_f with symbol f can be connected to the identity through a path in the invertible group of* AlgN, *where $f = \mathrm{e}^h$.*

2 Proofs of Theorems

Proof of Theorem 1.1 Since $\sigma(T)$ consists of finite points, we assume that

$$\sigma(T) = \{\lambda_i : i = 1, 2, \cdots, k\}.$$

For each i, $1 \leq i \leq k$, we can find an open neighborhood Ω_i of λ_i such that

$$\{\bar{\Omega}_i : i = 1, 2, \cdots, k\}$$

are pairwise disjoint. Set

$$\Omega = \bigcup_{i=1}^{k} \Omega_i.$$

For each i, $1 \leq i \leq k$, define

$$f_i(z) = \begin{cases} 1, & z \in \Omega_i; \\ 0, & z \in \Omega \setminus \Omega_i. \end{cases}$$