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The Almost Split Sequences for Trivial Extensions of Hereditary Algebras

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Abstract: Let A be a basic hereditary artin algebra and $R = A \ltimes Q$ be the trivial extension of A by its minimal injective cogenerator Q. We construct some right (left) almost split morphisms and irreducible morphisms in mod$R$ through the corresponding morphisms in mod$A$. Furthermore, we can determine its almost split sequences in mod$R$.

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1 Introduction

Let A be a basic hereditary artin algebra over its center C, I the injective envelope of $C/\mathrm{Rad}C$ and Q the A-module $\mathrm{Hom}_C(A, I)$. Then the trivial extension R of A by Q is $R = A \ltimes Q$. It is an additive group with the multiplication defined by

$$(a,\, q)(a',\, q') = (aa',\, aq' + qa'), \qquad (a, q), (a', q') \in R.$$

From [1], we know the following conclusion about the module category of $R = A \ltimes Q$.

Theorem 1.1[1] *Let A be a ring and M an A-module. Then the categories* $\mathrm{mod}A \ltimes M$, $\mathrm{mod}A \ltimes (M \otimes_A \_)$ *and* $\mathrm{Hom}_A(\_\,, M) \rtimes \mathrm{mod}A$ *are isomorphic.*

By the above theorem we know that the modules of $R = A \ltimes Q$ are in the forms of $(X \otimes_A Q \stackrel{\psi}{\to} X)$ or $(X \stackrel{\phi}{\to} \mathrm{Hom}_A(Q, X))$ for $X \in \mathrm{mod}A$. For the convenience, we just write $\mathrm{Hom}_A(Q, \_\,)$ as $[Q, \_\,]$. Since A is a hereditary algebra, $X = XQ \oplus X/XQ$.

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Let $U = X/XQ$ and $V = XQ$. Then the above forms have the canonical expressions $X = (U \otimes_A Q \stackrel{\psi}{\to} V)$ and $X = (U \stackrel{\phi}{\to} [Q, V])$. Since A is hereditary, V is an injective A-module from its construction (see [2]). In this paper we mainly consider the modules in the first form, i.e., $X = (U \otimes_A Q \stackrel{\psi}{\to} V)$. In order to study the irreducible morphisms, it is essential to determine the indecomposable modules of $R = A \ltimes Q$ which has been researched in [2].

Proposition 1.1[2] *Let A be a hereditary artin algebra, $Q = \mathrm{Hom}_C(A, I)$ and $R = A \ltimes Q$. Let $(U \otimes_A Q \stackrel{\psi}{\to} V) \in \mathrm{mod}A \ltimes (\_ \otimes_A Q)$ be a canonical expression of an R-module X with $\psi \neq 0$. Then X is an indecomposable R-module if and only if either of the following conditions* (1) *and* (2) *is satisfied*:

(1) *ψ is an isomorphism and U_A is indecomposable and projective*;

(2) *ψ is an epimorphism* (*but not monomorphism*), *U_A is projective,* $\ker\psi$ *is a large submodule of $U \otimes_A Q$ and is indecomposable.*

In case (1) *X is a projective and injective R-module.*

An indecomposable R-module $X = (U \otimes_A Q \stackrel{\psi}{\to} V)$ $((U \stackrel{\phi}{\to} [Q, V]))$ is call it to be of 2nd kind (1st kind) if $\psi \neq 0$ $(\psi = 0)$, and is same as $\phi \neq 0$ $(\phi = 0)$. There are also some consequences about R which can be seen in [3–4].

By the above theorem, proposition and the consequences about right (left) almost split morphisms and irreducible morphisms in [5–6], we construct the corresponding morphisms in mod$R$.

To begin with the discussion we recall the description of morphisms between R-modules. Let $X = (U \otimes_A Q \stackrel{\psi}{\to} V)$ and $X' = (U' \otimes_A Q \stackrel{\psi'}{\longrightarrow} V')$ be R-modules. Any R-morphism from X to X' has the matrix expression $\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$, where $\alpha : U \to U'$, $\beta : V \to V'$, $\gamma : V \to U'$ and $\delta : U \to V'$ are A-morphisms. At the same time, α and β are compatible with the following diagram:

$$\begin{array}{ccc} U \otimes Q & \xrightarrow{\psi} & V \\ {\scriptstyle \alpha\otimes 1_Q}\downarrow & & \downarrow{\scriptstyle \beta} \\ U' \otimes Q & \xrightarrow{\psi'} & V' \end{array} \tag{1.1}$$

From the construction of U' and V, we get that $\gamma = 0$.

All algebras in this paper are hereditary algebras and the tensor products are in the algebra A.

2 The Almost Split Morphisms over $R = A \ltimes Q$

In this section, we construct some right (left) almost split morphisms over $R = A \ltimes Q$ from the ones in mod$A$.