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A Split Least-squares Characteristic Procedure for Convection-dominated Parabolic Integro-differential Equations

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Abstract: In this paper, we combine a split least-squares procedure with the method of characteristics to treat convection-dominated parabolic integro-differential equations. By selecting the least-squares functional properly, the procedure can be split into two independent sub-procedures, one of which is for the primitive unknown and the other is for the flux. Choosing projections carefully, we get optimal order $H^1(\Omega)$ and $L^2(\Omega)$ norm error estimates for u and sub-optimal $(L^2(\Omega))^d$ norm error estimate for $\boldsymbol{\sigma}$. Numerical results are presented to substantiate the validity of the theoretical results.

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1 Introduction

We consider the following convection-dominated parabolic integro-differential equations:

$$\begin{cases} c(x)\dfrac{\partial u}{\partial t} + \boldsymbol{d}(x)\cdot\nabla u - \nabla\cdot\Big(\boldsymbol{A}(x)\nabla u + \boldsymbol{B}(x)\displaystyle\int_0^t \nabla u(x,s)\mathrm{d}s\Big) = f(x,t), & (x,t)\in(\Omega\times I),\\ u(x,t) = 0, & (x,t)\in(\Gamma\times I),\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $I=(0,T]$ is the time interval, Ω is a bounded polygonal domain in $\mathbf{R}^d$, $d=2,3$, with a Lipschitz continuous boundary Γ, d is the space dimension. $\boldsymbol{d}(x)=(d_1(x),\cdots,d_d(x))^{\mathrm{T}}$.

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$\boldsymbol{A}(x) = (a_{ij}(x))_{d\times d}$, $\boldsymbol{B}(x) = (b_{ij}(x))_{d\times d}$ are bounded, symmetric and positive definite matrices, i.e., there exist positive constants a_*, a^*, b_* and b^* such that

$$a_*\|\boldsymbol{\xi}\|^2 \le (\boldsymbol{A}\boldsymbol{\xi},\, \boldsymbol{\xi}) \le a^*\|\boldsymbol{\xi}\|^2, \quad b_*\|\boldsymbol{\xi}\|^2 \le (\boldsymbol{B}\boldsymbol{\xi},\, \boldsymbol{\xi}) \le b^*\|\boldsymbol{\xi}\|^2, \qquad \boldsymbol{\xi} \in \mathbf{R}^d.$$

We make the following assumptions: there exist positive constants k_1, k_2 such that

$$0 < k_1 \le c(x) \le k_2, \qquad \|\boldsymbol{d}\|_{1,\infty} + \|c\|_{1,\infty} \le k_2.$$

We also assume that Ω is H^2-regular: for $f \in L^2(\Omega)$ the solution of the following problem

$$-\nabla \cdot (\boldsymbol{A}\nabla w) = f \quad \text{in } \Omega, \qquad w|_\Gamma = 0$$

exists and $\|w\|_2 \le K\|f\|$.

This model arises from many physical processes in which it is necessary to take into account the effects of memory due to the deficiency of the usual diffusion equations (see [1–2]). As we all know, the numerical simulation of convection-dominated problems requires special treatment. Generally, they either smear sharp physical fronts with excessive numerical diffusion, or introduce nonphysical oscillations into numerical solutions. The method of characteristic has proved effective in treating convection-dominated problems (see [3–4]).

We have introduced the least-squares method for such equations when A, B are proportional to a unit matrix in [5]. The least-squares finite element procedure has two typical advantages as follows: it is not subject to Ladyzhenskaya[6], Babuška[7], Brezzi[8] consistency condition, so the choice of approximation spaces becomes flexible, and it results in a symmetric positive definite system. However, it usually needs to solve a coupled system of equations for conventional least-squares finite element procedure, which brings to difficulties in some extent. We only get the optimal order $H^1(\Omega)$ norm error estimate for u in [5].

In [9–10], a kind of split least-squares Galerkin procedure was constructed for stationary diffusion reaction problems and parabolic problems. The purpose of this paper is to combine the split least-squares procedure with the method of characteristics for convection-dominated parabolic integro-differential equations. The most advantage of the scheme is: by selecting the least-squares functional properly, the resulting procedure can be split into two independent symmetric positive definite sub-schemes. One of sub-procedures is for the primitive unknown variable u, which is the same as a stand Galerkin characteristic finite element procedure and the other is for the introduced flux variable $\boldsymbol{\sigma}$. By carefully choosing projections, we see that the method leads to the optimal order $H^1(\Omega)$ and $L^2(\Omega)$ norm error estimates for u and sub-optimal $(L^2(\Omega))^d$ norm error estimate for $\boldsymbol{\sigma}$.

The paper is organized as follows. In Section 2, we formulate the procedure. The convergence theory on the algorithm is established in Section 3. In Section 4 we give the numerical experiment.

As in [11], we assume that the problem (1.1) is periodic with Ω. In this paper we use $W^{k,p}$ $(k \ge 0,\ 1 \le p \le \infty)$ to denote Sobolev spaces (see [12]) defined on Ω with a usual norm $\|\cdot\|_{W^{k,p}(\Omega)}$, and $H^k(\Omega)$, $L^2(\Omega)$ with norms $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, respectively. For simplicity we also use $L^s(H^k)$ to denote $L^s(0,\, T;\, H^k(\Omega))$. The inner product $(\,\cdot\,,\,\cdot\,)$ is both used for scalar-valued functions and vector-valued functions. Throughout this paper, the symbols K and δ are used to denote a generic constant and a generic small positive constant, respectively, which may appear differently at different occurrences.