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Derivative Estimates for the Solution of Hyperbolic Affine Sphere Equation

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Abstract: Considering the hyperbolic affine sphere equation in a smooth strictly convex bounded domain with zero boundary values, the sharp derivative estimates of any order for its convex solution are obtained.
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1 Introduction

In affine differential geometry, the classification of complete hyperbolic affine hyperspheres has attracted the attention of many geometers. By a Legendre transformation, the classification of Euclidean-complete hyperbolic hyperspheres is reduced to the study of the following boundary value problem

$$\begin{cases} \det\left(\dfrac{\partial^2 u(x)}{\partial x_i \partial x_j}\right) = (-u(x))^{-n-2} & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded convex domain. Calabi[1] conjectured that there is a unique convex solution to (1.1). Loewner and Nirenberg[2] solved (1.1) in the cases of domains in $\mathbf{R}^2$ with smooth boundary. Cheng and Yau[3] showed there always exists a convex solution $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$, and the uniqueness follows from the maximum principal.

When $\Omega = B^n(1)$, the unit ball in $\mathbf{R}^n$, the convex solution of (1.1) is

$$u_0 = -\sqrt{1 - \sum_{1 \le k \le n} x_k^2}. \tag{1.2}$$

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When Ω is projectively homogeneous, Sasaki[4] found that the convex solution of (1.1) and the characteristic function χ of domain Ω have the following relation:

$$u = C_0 \chi^{-\frac{1}{n+1}} \qquad \text{for a constant } C_0.$$

Also, Sasaki and Yagi[5] obtained an expansion of derivatives of the characteristic function χ along the boundary of the smooth convex bounded domain. Referring the Fefferman's expansion of the Bergman kernel on smooth strictly pseudoconvex domains (see [6]), Sasaki[7] obtained an asymptotic expansion form of χ with respect to the solution u:

$$\chi = C_0 u^{-(n+1)} \Big[1 + \frac{5}{24(n-1)} F u^2 + \text{the higher orders of } u \Big], \tag{1.3}$$

where F is a smooth function on $\bar{\Omega}$.

In this paper, we confine ourselves to the case that Ω is a strictly convex bounded domain with smooth boundary. By the barrier functions on the balls, the convex solution of (1.1) has the bound:

$$\frac{1}{C} d(x)^{\frac{1}{2}} \le -u(x) \le C d(x)^{\frac{1}{2}}, \tag{1.4}$$

where $d(x) =: \text{dist}(x, \partial\Omega)$, and C is a positive constant depending on Ω and n.

By (1.4) and the convexity of u, the gradient estimate is given by:

$$\frac{1}{C} d(x)^{-\frac{1}{2}} \le |\text{grad}\, u| \le C d(x)^{-\frac{1}{2}}. \tag{1.5}$$

Loewner and Nirenberg[2] first obtained the sharp second order estimates in dimension two. Their methods and Pogorelov's calculations also gave bound for the higher dimensions (see [8]):

$$|u_{ij}| \le C d(x)^{-\frac{3}{2}}, \qquad 1 \le i, j \le n. \tag{1.6}$$

Now we introduce the basic notations. For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where α_i, $i = 1, 2, \cdots, n$, are non-negative integers with $|\alpha| = \sum\limits_{1 \le i \le n} \alpha_i$, we define

$$D_i = \frac{\partial}{\partial x_i}, \quad D_i^{\alpha_i} = \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}, \qquad i = 1, 2, \cdots, n,$$

$$D^{\alpha} = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

In this paper, by the finite geometry of complete hyperbolic affine sphere as stated in Lemma 2.1, we obtain derivative estimates of any order:

Theorem 1.1 *For $n = 2$, the convex solution of* (1.1) *satisfies*

$$|D^{\boldsymbol{\alpha}}(u)| \le C d(x)^{\frac{1}{2} - |\alpha|}, \qquad |\alpha| = 0, 1, 2, \cdots, \tag{1.7}$$

where C is a constant depending on Ω and $|\alpha|$.

Remark 1.1 For $|\alpha| = 3$, the estimate (1.7) holds for any dimension $n \ge 2$. The sharpness of exponent "$\frac{1}{2} - |\alpha|$" can be seen in the case that Ω is projectively homogeneous (see [5]).