Communications
in
Mathematical
Research
31(1)(2015), 71–80

Rota-Baxter Operators of Weight 1 on 2×2-matrix Algebras

Li Jin-zhi and Liu Wen-de*
(*School of Mathematical Sciences, Harbin Normal University, Harbin, 150025*)

Communicated by Du Xian-kun

Abstract: In this paper we determine all Rota-Baxter operators of weight 1 on 2×2-matrix algebras over an algebraically closed field.

Key words: Rota-Baxter operator, weight, associative algebra
2010 MR subject classification: 16A06, 47B99
Document code: A
Article ID: 1674-5647(2015)01-0071-10
DOI: 10.13447/j.1674-5647.2015.01.08

1 Introduction

Rota-Baxter algebras or their corresponding Rota-Baxter relations came from [1], which are on integral equations of fluctuation theory. After that, many mathematicians paid attention to this concept, and especially, Rota[2] fundamental papers around 1970 brought the subject into the areas of combinatorics and algebras. In fact, Rota-Baxter relation may be regarded as one possible generalization of the standard shuffle relation in [2–3]. In the case of Lie algebra, when the weight $\lambda = 0$, the Rota-Baxter relation is just the form of classical Yang-Baxter equation (CYBE) and when the weight $\lambda = 1$, it corresponds to the operator form of the modified classical Yang-Baxter equations. The broad connections of Rota-Baxter algebras with many areas in mathematics and mathematical physics are remarkable. However, the theoretical study of Rota-Baxter algebras is still in its early stage of development. In recent years, some articles have been published about Rota-Baxter algebras in [3–8]. Especially, An and Bai[7] have not only worked over Rota-Baxter operators of weight 0 on pre-Lie algebras, but also computed all Rota-Baxter operators of weight 0 on associative algebras of dimensions ≤ 3; Li and Hou[8] have given all Rota-Baxter operators of weight 1 on associative algebras of dimensions ≤ 3.

In this paper, we study the Rota-Baxter operators of weight 1 on the associative algebra

Received date: Jan. 7, 2013.
Foundation item: The NSF (JC201004) of Heilongjiang Province and the NSF (11171055) of China.
* **Corresponding author**.
E-mail address: lijinzhi800213@163.com (Li J Z), wendeliu@ustc.edu.cn (Liu W D).

$M_2(\mathbb{F})$ consisting of 2×2-matrix over an algebraically closed field $\mathbb{F}$. The paper is organized as follows. In Section 2, we give all Rota-Baxter operators of weight 1 on $M_2(\mathbb{F})$. In Section 3, we analyze how to prove the previous theorem and establish the corresponding equations. In Section 4, we give the proof of the main theorem. Throughout this paper, all algebras are of finite dimensions and over an algebraically closed field $\mathbb{F}$.

2 Main Results

We adopt the following definition from [3].

Definition 2.1 *A Rota-Baxter algebra is an associative algebra A over $\mathbb{F}$ with a linear operator $R : A \to A$ satisfying the Rota-Baxter relation*:

$$R(x)R(y) + \lambda R(xy) = R(R(x)y + xR(y)), \qquad x, y \in A, \tag{2.1}$$

where $\lambda \in \mathbb{F}$ is a fixed element, which is called the weight of R.

If R is a Rota-Baxter operator of weight $\lambda \neq 0$, then $\frac{1}{\lambda}R$ is a Rota-Baxter operator of weight 1. Thus, for the Rota-Baxter operators of nonzero weight, it suffices to determine the ones of weight 1. From now on, $RB(A)$ denotes the set of all Rota-Baxter operators on A of weight 1.

Theorem 2.1 *All the Rota-Baxter operators of weight 1 on 2×2-matrix algebra $M_2(\mathbb{F})$ are R_i and $I - R_i \in RB(M_2(\mathbb{F}))$, $i = 1, 2, \cdots, 31$, shown in Table 2.1, where $a, b, c \in \mathbb{F}$ (any denominator is nonzero and non-appeared parameters in matrices are zero).*

Table 2.1 The Rota-Baxter operators set

Operators number	Matrix representations of operators
1	$r_{13} = -r_{21} = a$, $r_{14} = -r_{22} = b$, $r_{33} = -r_{41} = 1 + b$, $r_{34} = -r_{42} = \dfrac{b + b^2}{a}$
2	$r_{11} = r_{33} = a$, $r_{21} = r_{43} = b$, $r_{22} = r_{44} = 1 - a$, $r_{12} = r_{34} = \dfrac{a - a^2}{b}$
3	$r_{11} = -r_{41} = a$, $r_{21} = b$, $r_{22} = 1 - a$, $r_{12} = -r_{42} = \dfrac{a - a^2}{b}$, $r_{31} = -\dfrac{a^2}{b}$, $r_{32} = \dfrac{a^3 - a^2}{b}$
4	$r_{11} = a$, $r_{21} = b$, $r_{22} = r_{41} = 1 - a$, $r_{12} = r_{31} = -r_{42} = \dfrac{a - a^2}{b}$, $r_{32} = \dfrac{a^3 - a^2}{b^2}$, $r_{34} = \dfrac{a}{b}$, $r_{44} = 1$
5	$r_{11} = r_{22} = a$, $r_{13} = r_{24} = b$, $r_{31} = r_{42} = \dfrac{a - a^2}{b}$, $r_{33} = r_{44} = 1 - a$
6	$r_{14} = -r_{33} = a$, $r_{24} = -r_{43} = b$, $r_{22} = -r_{41} = 1 + a$, $r_{12} = -r_{31} = \dfrac{a + a^2}{b}$
7	$r_{14} = -r_{44} = a$, $r_{22} = 1 + a$, $r_{24} = b$, $r_{12} = -r_{34} = -r_{42} = \dfrac{a + a^2}{b}$, $r_{31} = \dfrac{1 + a}{b}$, $r_{33} = 1$, $r_{32} = \dfrac{-a(1 + a)^2}{b^2}$

(to be continued)