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Additive Maps Preserving the Star Partial Order on $\mathcal{B}(\mathcal{H})$

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Abstract: Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. It is proved that an additive surjective map φ on $\mathcal{B}(\mathcal{H})$ preserving the star partial order in both directions if and only if one of the following assertions holds. (1) There exist a nonzero complex number α and two unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that $\varphi(\boldsymbol{X}) = \alpha \boldsymbol{UXV}$ or $\varphi(\boldsymbol{X}) = \alpha \boldsymbol{UX}^* \boldsymbol{V}$ for all $\boldsymbol{X} \in \mathcal{B}(\mathcal{H})$. (2) There exist a nonzero α and two anti-unitary operators $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\mathcal{H}$ such that $\varphi(\boldsymbol{X}) = \alpha \boldsymbol{UXV}$ or $\varphi(\boldsymbol{X}) = \alpha \boldsymbol{UX}^* \boldsymbol{V}$ for all $\boldsymbol{X} \in \mathcal{B}(\mathcal{H})$.

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1 Introduction

In the last few decades, many researchers have studied properties of various partial orders on matrix algebras, or operator algebras acting on a complex infinite dimensional Hilbert space, such as minus partial order, star partial order, left and right star partial order and so on (see [1–6]). One of the orders on the algebra M_n of all $n \times n$ complex matrices is the star partial order "$\overset{*}{\leq}$" defined by Drazin in [5]. Let $\boldsymbol{A}, \boldsymbol{B} \in M_n$. Then we say that $\boldsymbol{A} \overset{*}{\leq} \boldsymbol{B}$ if $\boldsymbol{A}^*\boldsymbol{A} = \boldsymbol{A}^*\boldsymbol{B}$ and $\boldsymbol{AA}^* = \boldsymbol{BA}^*$. We note that this definition can be extended to a C^*-algebra by the same way. In particular, it can be extended to the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For example, motivated by Šemrl's approach presented in [7] for minus partial order, Dolinar and Marovt[4] gave an equivalent

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definition (see Definition 2 in [4]) of the star partial order and considered some properties of this partial order. We can refer [1, 4] to see more interesting properties.

On the other hand, as partially ordered algebraic structures on M_n and $\mathcal{B}(\mathcal{H})$, what are the automorphisms of M_n and $\mathcal{B}(\mathcal{H})$ with respect to those partial orders? These topics have been studied and some interesting results have been obtained. Šemrl[7] described the structure of corresponding automorphisms for the minus partial order. For the star partial order, Guterman[8] characterized linear bijective maps on M_n preserving the star partial order and Legiša[9] considered automorphisms of M_n with respect to the star partial order. Recently, several authors consider the automorphisms of certain subspaces of $\mathcal{B}(\mathcal{H})$ with respect to the star partial order when $\mathcal{H}$ is infinite dimensional. Dolinar and Guterman[10] studied the automorphisms of the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable complex Hilbert space $\mathcal{H}$ and they characterized the bijective, additive, continuous maps on $\mathcal{K}(\mathcal{H})$ which preserve the star partial order in both directions. On the other hand, characterizations of certain continuous bijections on the normal elements of a von Neumann algebra preserving the star partial order in both directions are obtained by Bohata and Hamhalter[11]. In this paper, we consider additive surjective maps preserving the star partial order in both directions on $\mathcal{B}(\mathcal{H})$ and characterizations of those maps are given. In particular, we improve the main result in [10].

Let $\mathcal{H}$ be a complex Hilbert space and denote by $\dim\mathcal{H}$ the dimension of $\mathcal{H}$. Let $\mathbb{C}$ and $\mathbb{Q}$ denote the complex field and the rational number field, respectively. Let $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$ be the algebras of all bounded linear operators, the compact operators and the finite rank operators on $\mathcal{H}$, respectively. For every pair of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ denotes the inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{x} \otimes \boldsymbol{y}$ stands for the rank-1 linear operator on $\mathcal{H}$ defined by $(\boldsymbol{x} \otimes \boldsymbol{y})\boldsymbol{z} = \langle \boldsymbol{z}, \boldsymbol{y}\rangle \boldsymbol{x}$ for any $\boldsymbol{z} \in \mathcal{H}$. If $\boldsymbol{x}$ is a unit vector, then $\boldsymbol{x} \otimes \boldsymbol{x}$ is a rank-1 projection. $\sigma(\boldsymbol{A})$ is the spectrum of $\boldsymbol{A}$ for any $\boldsymbol{A} \in \mathcal{B}(\mathcal{H})$. For a subset S of $\mathcal{H}$, $[S]$ denotes the closed subspace of $\mathcal{H}$ spanned by S and $\boldsymbol{P}_M$ denotes the orthogonal projection on M for a closed subspace M of $\mathcal{H}$. We denote by $R(\boldsymbol{T})$ and $N(\boldsymbol{T})$ the range and the kernel of a linear map $\boldsymbol{T}$ between two linear spaces. Throughout this paper, we generally denote by $\boldsymbol{I}$ the identity operator on a Hilbert space.

2 Additive Maps Preserving the Star Partial Order

Let φ be an additive map on $\mathcal{B}(\mathcal{H})$. We say that φ preserves the star partial order if $\varphi(\boldsymbol{A}) \overset{*}{\leq} \varphi(\boldsymbol{B})$ for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}(\mathcal{H})$ such that $\boldsymbol{A} \overset{*}{\leq} \boldsymbol{B}$. We say that φ preserves the star partial order in both directions if $\varphi(\boldsymbol{A}) \overset{*}{\leq} \varphi(\boldsymbol{B})$ if and only if $\boldsymbol{A} \overset{*}{\leq} \boldsymbol{B}$ for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{B}(\mathcal{H})$. We firstly give the following lemma which generalizes Lemma 10 in [10].

Let $\boldsymbol{T} \in \mathcal{B}(\mathcal{H})$. We denote by

$$H_1 = \overline{R(\boldsymbol{T}^*)}, \qquad H_2 = N(\boldsymbol{T}), \qquad K_1 = \overline{R(\boldsymbol{T})}, \qquad K_2 = N(\boldsymbol{T}^*),$$

respectively. Then

$$\mathcal{H} = H_1 \oplus H_2 = K_1 \oplus K_2, \tag{2.1}$$