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Nonlinear Jordan Higher Derivations of Triangular Algebras

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Abstract: In this paper, we prove that any nonlinear Jordan higher derivation on triangular algebras is an additive higher derivation. As a byproduct, we obtain that any nonlinear Jordan derivation on nest algebras over infinite dimensional Hilbert spaces is inner.

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1 Introduction

Let $\mathcal{R}$ be a commutative ring with identity and let $\mathcal{A}$ be an $\mathcal{R}$-algebra. We denote by $x \circ y = xy + yx$ the Jordan product of two elements $x, y \in \mathcal{A}$. An $\mathcal{R}$-linear map $d : \mathcal{A} \to \mathcal{A}$ is called a Jordan derivation if $d(x \circ y) = d(x) \circ y + x \circ d(y)$ for all $x, y \in \mathcal{A}$. If the linear assumption in the above definition is deleted, the corresponding map is said to be a nonlinear Jordan derivation. It should be remarked that there are several definitions of (linear) Jordan derivations and all of them are equivalent as long as the algebra $\mathcal{A}$ is 2-torsion free. We refer the reader to [1] for more details and related topics.

In 1957, Herstein[2] proved that every Jordan derivation on a 2-torsion free prime ring is a derivation. After that, it is of much interest for researchers to find which algebras could make a Jordan derivation degenerate to a derivation. Cusack[3] and Brešar[4] independently found out that the 2-torsion free semiprime rings satisfy the condition. Recently, Johnson[5] proved that a continuous Jordan derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is a derivation.

Let A and B be two unital algebras over $\mathcal{R}$ and $\mathcal{M}$ a unital (A, B)-bimodule which is faithful as a left A-module and also as a right B-module. A triangular algebra $\mathcal{T}$ is an

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associative algebra of the form

$$\begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \middle| \ a \in A,\ b \in B,\ m \in M \right\}$$

under a matrix-like addition and a matrix-like multiplication.

Let us now recall some basic facts related to Jordan higher derivations of an associative algebra. Many different kinds of higher derivations, which consist of a family of some additive mappings, have been widely studied in commutative and noncommutative rings. Let $\mathbf{N}$ be the set of all non-negative integers and $D = \{d_n\}_{n\in\mathbf{N}}$ be a family of $\mathcal{R}$-linear mappings on $\mathcal{A}$ such that $d_0 = \mathrm{id}_{\mathcal{A}}$. D is called:

(a) a higher derivation if

$$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$$

for all $x, y \in \mathcal{A}$ and for each $n \in \mathbf{N}$;

(b) a Jordan higher derivation if

$$d_n(x \circ y) = \sum_{i+j=n} d_i(x) \circ d_j(y)$$

for all $x, y \in \mathcal{A}$ and for each $n \in \mathbf{N}$.

As far as we know, Cheung[6–7] first initiated the study of linear maps on triangular algebras. Then Zhang and Yu[8] proved that every Jordan derivation on a 2-torsion free triangular algebra is a derivation. This result was extended to Jordan higher derivations by Xiao and Wei[9]. Most recently, Yu and Zhang[10] generalized the main results of [7, 11] to nonlinear Lie derivations and then Xiao and Wei[12] extended to nonlinear Lie higher derivations. Motivated by the above works, we investigate nonlinear Jordan higher derivations, analogous to nonlinear Lie higher derivations, on triangular algebras.

2 Nonlinear Jordan Derivations of Triangular Algebras

In this section, we first study nonlinear Jordan derivations on triangular algebras and nest algebras. From now on, without claim we always assume that any algebra and (bi-)module considered are 2-torsion free.

Let $\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be a triangular algebra. Let 1_A and 1_B be the identity of the algebra A and B, respectively, and I be the identity of $\mathcal{T}$. Denote

$$P = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q = I - P = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix},$$

and

$$\mathcal{T}_{11} = P\mathcal{T}P, \qquad \mathcal{T}_{12} = P\mathcal{T}Q, \qquad \mathcal{T}_{22} = Q\mathcal{T}Q.$$

Thus the triangular algebra $\mathcal{T}$ can be expressed as

$$\mathcal{T} = \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{22}.$$

Clearly, $\mathcal{T}_{11}$ and $\mathcal{T}_{22}$ are subalgebras of $\mathcal{T}$ which are isomorphic to A and B, respectively. $\mathcal{T}_{12}$ is a $(\mathcal{T}_{11}, \mathcal{T}_{22})$-bimodule which is isomorphic to the (A, B)-bimodule M.

The main result of this section is as follows.