COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
31(3)(2015), 261–266

A Note on Zinbiel Algebras

NI JUN-NA
(*School of Mathematical Sciences, South China Normal University, Guangzhou, 510631*)

Communicated by Du Xian-kun

Abstract: In this paper, we consider the decomposition of Zinbiel algebras with a special bilinear form which are called quadratic Zinbiel algebras. We obtain that the decomposition into irreducible ideals of a quadratic Zinbiel algebra is unique under an isometry.

Key words: Zinbiel algebra, quadratic, decomposition, isometry
2010 MR subject classification: 17A30, 17A32
Document code: A
Article ID: 1674-5647(2015)03-0261-06
DOI: 10.13447/j.1674-5647.2015.03.08

1 Introduction

Lie algebra is one of the important object of the modern theory of non-associative algebras. Active investigations in the theory of Lie algebras lead to the appearing of some generalizations of these algebras such as Malcev algebras, Lie superalgebras, binary Lie algebras, Leibniz algebras and so on. Recall that Leibniz algebras are defined by the following identity

$$[x,\ [y, z]] = [[x, y],\ z] - [[x, z],\ y]. \tag{1.1}$$

The category of Zinbiel algebras is Koszul dual to the category of Leibniz algebras in the sense of Loday[1]. So Zinbiel algebras are also called dual Leibniz algebras (see [2]).

A Zinbiel algebra is a vector space A together with a bilinear map $\circ : A \times A \to A$ such that

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y). \tag{1.2}$$

One can get a commutative associative algebra structure on A under the new defined product $x \circ y + y \circ x$.

Zinbiel algebras have a close relationship with dendriform algebras, pre-Lie algebras, etc. (see [3–4]). We recall their definitions in the following.

Received date: Dec. 3, 2013.
E-mail address: nijunna@126.com (Ni J N).

A pre-Lie algebra is a vector space A together with a bilinear map $\cdot : A \times A \to A$ such that

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z. \tag{1.3}$$

It is well known that A becomes a Lie algebra under the new defined product $x \cdot y - y \cdot x$.

A dendriform algebra is a vector space A with two bilinear products denoted by $\prec$ and $\succ$. If for any $x, y, z \in A$, then the following equalities hold:

$$\begin{aligned}(x \prec y) \prec z &= x \prec (y * z);\\ (x \succ y) \prec z &= x \succ (y \prec z);\\ x \succ (y \succ z) &= (x * y) \succ z,\end{aligned} \tag{1.4}$$

where $x * y = x \prec y + x \succ y$. Define a new product $\star : A \times A \to A$ by $x \star y = x \prec y + x \succ y$, then A becomes an associative algebra.

As a commutative algebra is an associative algebra for which $x \circ y = y \circ x$, a Zinbiel algebra may be equivalently defined as a dendriform algebra for which $x \succ y = y \prec x$. If one thinks of dendrifrom algebras as an analog of associative algebras, then one views Zinbiel and pre-Lie algebras as the "dendriform analog" of commutative and Lie algebras, respectively. The situation is summarized by means of the following diagram, where one thinks of the rows as "exact sequences" (see [3, 5]):

$$\begin{array}{ccccc} \text{Zinbiel} & \to & \text{Dendrifrom} & \to & \text{Pre-Lie} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Commutative} & \to & \text{Associative} & \to & \text{Lie} \end{array}$$

Also Zinbiel algebra is concerned with pre-Poisson algebras. It is shown that the free Zinbiel algebra (the shuffle algebra) on a pre-Lie algebra is a pre-Poisson algebra (see [5]). So Zinbiel algebra is a kind of important algebra structure as well as associative or Lie algebras and so on. The study of Zinbiel algebras is obviously difficult due to the non-associativity. In spite of some progress (see [6–12]), there are still many open questions.

However, there is very few study about the decomposition on Zinbiel algebras. In this paper, we investigate the decomposition of Zinbiel algebras with a associative bilinear form. A bilinear form $f : A \times A \to \mathbf{C}$ is associative if and only if

$$f(xy,\, z) = f(x,\, yz), \qquad x, y, z \in A. \tag{1.5}$$

The goal of this paper is to study the pair $(A,\, f)$, where A denotes a Zinbiel algebra and f denotes a non-degenerate associative symmetric bilinear form on A. In abuse of notation we also use the term quadratic Zinbiel algebra for denoting such a pair. The motivation for studying quadratic Zinbiel algebras comes from the fact that Lie or associative algebras with forms have important applications in several areas of mathematics and physics, such as the structure theory of finite-dimensional semi-simple Lie algebras, the theory of complete integrable Hamiltonian systems and the classification of statistical models over two-dimensional graphs.

Throughout this paper, all algebras are finite-dimensional and over the complex field $\mathbf{C}$.