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# A Note on Weighted Composition Operators on the Fock Space

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**Abstract:** Based on a new characterization of bounded and compact weighted composition operators on the Fock space obtained by Le T (Le T. Normal and isometric weighted composition operators on the Fock space. *Bull. London. Math. Soc.*, 2014, **46**: 847–856), this paper shows that a bounded weighted composition operator on the Fock space is a Fredholm operator if and only if it is an invertible operator, and if and only if it is a nonzero constant multiple of a unitary operator. The result is very different from the corresponding results on the Hardy space and the Bergman space.

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## 1 Introduction

Recently, weighted composition operators on the Fock space have been studied in [1–8]. In [3], the boundedness and compactness of weighted composition operators on the Fock space are characterized explicitly, as corollaries, the normal and isometric weighted composition operators are characterized completely. The self-adjoint and unitary weighted composition operators and their spectrum on the Fock space are characterized in [7–8], respectively. In this paper, we study Fredholm weighted composition operators on the Fock space. It is amazing that a bounded weighted composition operator on the Fock space is a Fredholm operator if and only if it is an invertible operator, and if and only if it is a nonzero constant

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multiple of a unitary operator.

Recall that the Fock space $F^2$ is the space of entire functions $f$ on $\mathbf{C}$ satisfying

$$\|f\|^2 = \frac{1}{2\pi}\int_{\mathbf{C}} |f(z)|^2 \mathrm{e}^{-|z|^2} \mathrm{d}m(z),$$

where $\mathrm{d}m$ is the usual Lebesgue measure on $\mathbf{C}$. It is well known that $F^2$ is a reproducing kernel Hilbert space with reproducing kernel function

$$K_w(z) = \mathrm{e}^{\bar{w}z}, \qquad w, z \in \mathbf{C}.$$

Let $k_w$ be the normalization of $K_w$. Then

$$k_w(z) = \mathrm{e}^{\bar{w}z - \frac{|w|^2}{2}}.$$

For more information on the Fock spaces and their operators, see [9].

For an entire function $\varphi$ on $\mathbf{C}$ and $\psi \in F^2$, the weighted composition operator $C_{\psi,\varphi}$ on $F^2$ is defined as

$$C_{\psi,\varphi} f = \psi(f \circ \varphi), \qquad f \in F^2.$$

Our main result is as follows.

**Theorem 1.1** *Let $\varphi$ be an entire function on $\mathbf{C}$ and $\psi$ be a nonzero function in $F^2$. If $C_{\psi,\varphi}$ is bounded on $F^2$, then the following conditions are equivalent*:

(1) *$C_{\psi,\varphi}$ is a nonzero constant multiple of a unitary operator*;

(2) *$C_{\psi,\varphi}$ is an invertible operator*;

(3) *$C_{\psi,\varphi}$ is a Fredholm operator.*

# 2 Proof of the Main Result

Before proving Theorem 1.1, some known results are needed.

**Lemma 2.1** *Let $\varphi$ be an entire function on $\mathbf{C}$ and $\psi \in F^2$. If $C_{\psi,\varphi}$ is bounded on $F^2$, then*

$$C_{\psi,\varphi}^* K_w = \overline{\psi(w)} K_{\varphi(w)}.$$

**Lemma 2.2**[3] *Let $\varphi$ be an entire function on $\mathbf{C}$ and $\psi$ be a nonzero function in $F^2$.*

(1) *$C_{\psi,\varphi}$ is bounded on $F^2$ if and only if $\varphi(z) = az + b$ with $|a| \le 1$ and*

$$\sup_{z\in\mathbf{C}} |\psi(z)|^2 \mathrm{e}^{|\varphi(z)|^2 - |z|^2} < \infty;$$

(2) *$C_{\psi,\varphi}$ is compact on $F^2$ if and only if $\varphi(z) = az + b$ with $|a| < 1$ and*

$$\lim_{|z|\to\infty} |\psi(z)|^2 \mathrm{e}^{|\varphi(z)|^2 - |z|^2} = 0.$$

The following result follows from Propositions 2.1, 3.1 and Theorem 2.2 in [3].

**Lemma 2.3** *Let $\varphi(z) = az + b$ with $|a| = 1$ and $\psi$ be a nonzero function in $F^2$. If $C_{\psi,\varphi}$ is bounded on $F^2$, then $C_{\psi,\varphi}$ is a nonzero constant multiple of a unitary operator.*