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Co-commuting Mappings of Generalized Matrix Algebras

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Abstract: Let $\mathcal{G}$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$ and $Z(\mathcal{G})$ be the center of $\mathcal{G}$. Suppose that $F, T\colon \mathcal{G} \longrightarrow \mathcal{G}$ are two co-commuting $\mathcal{R}$-linear mappings, i.e., $F(x)x = xT(x)$ for all $x \in \mathcal{G}$. In this note, we study the question of when co-commuting mappings on $\mathcal{G}$ are proper.

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1 Introduction

Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ be a unital algebra over $\mathcal{R}$ and $Z(\mathcal{A})$ be the center of $\mathcal{A}$. Let us denote the commutator or the Lie product of the elements $a, b \in \mathcal{A}$ by $[a, b] = ab - ba$. Recall that an $\mathcal{R}$-linear mapping $f : \mathcal{A} \to \mathcal{A}$ is said to be commuting if $[f(a), a] = 0$ for all $a \in \mathcal{A}$. A commuting mapping f of $\mathcal{A}$ is called proper if

$$f(a) = ca + \theta(a), \qquad a \in \mathcal{A},$$

where $c \in Z(\mathcal{A})$ and θ is an $\mathcal{R}$-linear mapping from $\mathcal{A}$ to $Z(\mathcal{A})$. As a generalization of commuting mappings, a pair of $\mathcal{R}$-mappings $F, T : A \to A$ is said to be co-commuting if

$$F(a)a = aT(a), \qquad a \in \mathcal{A}.$$

A pair of co-commuting mappings F, T is called proper if

$$F(a) = ac + \theta(a), \quad T(a) = ca + \theta(a), \qquad a \in \mathcal{A},$$

where $c \in \mathcal{A}$ and θ is an $\mathcal{R}$-linear mapping from $\mathcal{A}$ to $Z(\mathcal{A})$.

The usual goal when treating a commuting mapping is to describe its form. Generally, the form of commuting mappings can describe some structure information of algebras, especially, on the commutativity. The most important result on commuting mappings is Posner's theorem which states that the existence of a nonzero commuting derivation on a

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prime algebra $\mathcal{A}$ implies that $\mathcal{A}$ is commutative (see [1]). Brešar[2] showed that commuting mappings on prime algebras are proper. We encourage the reader to read the well-written survey paper (see [3]), in which the author presented the development of the theory of commuting mappings and their applications in detail. The following topics are discussed in [3]: commuting derivations, commuting additive mappings, commuting traces of multiadditive mappings, various generalizations of the notion of a commuting mapping, and applications of results on commuting mappings to different areas, in particular to Lie theory. These topics have formed the theory of functional identities, which deals with mappings of rings satisfying some identical relations (see [4–12]).

Let us recall the definition of generalized matrix algebras given by a Morita context. Let $\mathcal{R}$ be a commutative ring with identity. A Morita context consists of two $\mathcal{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \underset{B}{\otimes} N \to A$ and $\Psi_{NM} : N \underset{A}{\otimes} M \to B$ satisfying the following commutative diagrams:

$$
\begin{array}{ccc}
M \underset{B}{\otimes} N \underset{A}{\otimes} M & \xrightarrow{\Phi_{MN}\otimes I_M} & A \underset{A}{\otimes} M \\
\Big\downarrow {\scriptstyle I_M\otimes\Psi_{NM}} & & \Big\downarrow {\scriptstyle \cong} \\
M \underset{B}{\otimes} B & \xrightarrow{\cong} & M
\end{array}
\qquad
\begin{array}{ccc}
N \underset{A}{\otimes} M \underset{B}{\otimes} N & \xrightarrow{\Psi_{NM}\otimes I_N} & B \underset{B}{\otimes} N \\
\Big\downarrow {\scriptstyle I_N\otimes\Phi_{MN}} & & \Big\downarrow {\scriptstyle \cong} \\
N \underset{A}{\otimes} A & \xrightarrow{\cong} & N
\end{array}
$$

We immediately obtain a matrix algebra defined by the above Morita context (see [13]) and denoted by $\begin{bmatrix} A & M \\ N & B \end{bmatrix}$. Such an $\mathcal{R}$-algebra $\boldsymbol{\mathcal{G}} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ is called a generalized matrix algebra if M is faithful as a left A-module and also as a right B-module. If $N = 0$, then $\boldsymbol{\mathcal{G}}$ exactly degenerates to the so-called triangular algebra. We refer the reader to Section 2 in [14] for more examples of generalized matrix algebra.

Cheung[15–16] initiated the study of commuting mappings of matrix algebras and determined the class of triangular algebras for which every commuting linear mapping is proper. Benkovič and Eremita[17] studied commuting traces of bilinear mappings on triangular algebras, and gave the conditions under which every commuting trace of a triangular algebra is proper. This is applied to the study of Lie isomorphisms and of commutativity preserving mappings. Xiao and Wei[14] and their cooperators dealt with commuting mappings, semicentralizing mappings (see [18]) on a generalized matrix algebra. Following their steps, we now study the question of when co-commuting mappings on a generalized matrix algebra $\boldsymbol{\mathcal{G}}$ are proper.

2 Co-commuting Maps of Generalized Matrix Algebras

In this section, we characterize the co-commuting mappings of the generalized matrix algebras $\boldsymbol{\mathcal{G}} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$, where M is faithful as a left A-module and also as a right B-module.