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L_p-centroid Bodies and Its Characterizations

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Abstract: In this paper, we study the characteristic properties for L_p-centroid bodies, and an improved version of Busemann-Petty problem for L_p-centroid bodies is obtained. In addition, using the definitions of L_p-pole curvature image and L_p-affine surface area, a new proof of Busemann-Petty problem for L_p-centroid bodies is given.
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1 Introduction

The concept of classic centroid body was first proposed by Blaschke and Dupin (see [1]), and was defined by Petty[2]. Lutwak and Zhang[3] introduced the concept of L_p-centroid body. For each convex subset in $\mathbf{R}^n$, it is well-known that there is a unique ellipsoid with the following property: The moment of inertia of the ellipsoid and the moment of inertia of the convex set are the same about every 1-dimensional subspace of $\mathbf{R}^n$. This ellipsoid is called the Lengendre ellipsoid of the convex set. Namely, L_2-centroid body $\Gamma_2 K$. The Lengendre ellipsoid and its polar (the Binet ellipsoid) are well-known concepts from classical mechanics.

As usual, $V(K)$ denotes the n-dimensional volume of a body K in Euclidean space $\mathbf{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$. Let B denote the centered (centrally symmetric with respect to the origin) unit ball in $\mathbf{R}^n$, and we write $\omega_n = V(B)$ for its volume.

The definition of the classic centroid body was introduced by Petty[2]: Let K be a star body (about the origin) in $\mathbf{R}^n$. Then the classic centroid body, ΓK, of K is the origin-

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symmetric convex body whose support function is given by

$$h_{\Gamma K}(\boldsymbol{u}) = \frac{1}{V(K)} \int_K |\boldsymbol{u} \cdot \boldsymbol{x}| \mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{u} \in S^{n-1},$$

where $\boldsymbol{x} \cdot \boldsymbol{y}$ denotes the standard inner product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbf{R}^n$.

The classic centroid body is an important concept in convex geometry analysis. About the research of classic centroid body, Petty[4], Lutwak[5−7] and Zhang[8−10] have made plentiful and substantial achievements.

In 1997, Lutwak and Zhang[3] introduced the notion of L_p-centroid body, which extend the concept of the classical centroid body. Let K be a star body (about the origin) in $\mathbf{R}^n$ and $p \geq 1$. Then the L_p-centroid body, $\Gamma_p K$, of K is the origin-symmetric convex body whose support function is given by

$$h_{\Gamma_p K}(\boldsymbol{u}) = \frac{1}{c_{n,p} V(K)} \int_K |\boldsymbol{u} \cdot \boldsymbol{x}|^p \mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{u} \in S^{n-1}, \tag{1.1}$$

where

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}, \qquad \omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}.$$

By using polar coordinate transformation in (1.1), we can obtain

$$h^p_{\Gamma_p K}(\boldsymbol{u}) = \frac{1}{(n+p) c_{n,p} V(K)} \int_{S^{n-1}} |\boldsymbol{u} \cdot \boldsymbol{v}|^p \rho_K^{n+p}(\boldsymbol{v}) \mathrm{d}S(\boldsymbol{v}), \qquad \boldsymbol{u} \in S^{n-1}. \tag{1.2}$$

For set of the palar bodies of all L_p-projection bodies, we define the following:

$\Pi_p^* = \{\Pi_p^* Q :\ Q \subset \mathbf{R}^n$ is the any convex body containing origin in their interiors$\}$,

where $\Pi_p^* Q$ denotes the polar body of L_p-projection body $\Pi_p Q$. The following results are equivalent:

(i) $L \in \Pi_p^*$;

(ii) $(\mathbf{R}^n, ||\cdot||_L)$ is isometric to a subspace of L_p.

The latter fact can be found in [11].

Concerning the operator Γ_p, a well-known Shephard problem can be stated as follows: Let K, L be two origin-symmetric convex bodies in $\mathbf{R}^n$ and suppose that, for every $p \geq 1$, $\Gamma_p K \subseteq \Gamma_p L$, does it follow that we have an inequality for the volumes of K and L?

The premier solution of this problem was given by Grinberg and Zhang[12]. Their results are described as follows:

Theorem 1.1 *Let K be a star body* (*about the origin*) *in $\mathbf{R}^n$, $L \in \Pi_p^*$, and $p \geq 1$. If $\Gamma_p K \subseteq \Gamma_p L$, then*

$$V(K) \leq V(L)$$

with equality for $n \neq p \geq 1$ if and only if $K = L$.

On the other hand, if $K \notin \Pi_p^*$, then there is a body L such that $\Gamma_p K \subseteq \Gamma_p L$, but $V(K) > V(L)$.

In this paper, we further study the characteristic properties of L_p-centroid bodies. First, we give the following result for L_p-centroid bodies.