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Stability of Semi-implicit Finite Volume Scheme for Level Set Like Equation

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Abstract: We study numerical methods for level set like equations arising in image processing and curve evolution problems. Semi-implicit finite volume-element type schemes are constructed for the general level set like equation (image selective smoothing model) given by Alvarez *et al.* (Alvarez L, Lions P L, Morel J M. Image selective smoothing and edge detection by nonlinear diffusion II. *SIAM J. Numer. Anal.*, 1992, **29**: 845–866). Through the reasonable semi-implicit discretization in time and co-volume method for space approximation, we give finite volume schemes, unconditionally stable in L_∞ and $W^{1,2}$ ($W^{1,1}$) sense in isotropic (anisotropic) diffusion domain.

Key words: level set like equation, semi-implicit, finite volume scheme, stability

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1 Introduction

Level set like nonlinear parabolic equations arise in a wide range of applications as image processing and computer vision, phase transition, crystal growth, flame propagation, etc., and they are mainly related to the curve and surface evolution such as mean curvature motion of level set. In this paper, we study numerical methods for solving the following general level set like equation (image selective smoothing model combining isotropic and anisotropic diffusion) given by Alvarez *et al.*[1]:

$$u_t - g(|\nabla u_\sigma|)\Big(\big(1 - f(|\nabla u|)\big)\Delta u + f(|\nabla u|)|\nabla u|\mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big)\Big) = 0, \tag{1.1}$$

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where $u(t,x)$ is a unknown function defined in $Q \equiv I \times \Omega$. Here $\Omega \in \mathbf{R}^d$ is assumed to be a bounded rectangular domain (such an assumption coincides with the purpose for image processing) and $I = [0,T]$ is a time (scale) interval. Assume that $g : \mathbf{R}_0^+ \to \mathbf{R}^+$ is a nonincreasing function, $g(\sqrt{s})$ is smooth. In the image processing applications, we assume that $g(0) = 1$ and $g(s) \to 0$ as $s \to \infty$.

$$u_\sigma := G_\sigma * u = \int_{\mathbf{R}^d} G_\sigma(x-\xi)\tilde{u}(\xi)\mathrm{d}\xi,$$

where $G_\sigma \in C^\infty(\mathbf{R}^d)$ is a smoothing kernel with $\int_{\mathbf{R}^d} G_\sigma(x)\mathrm{d}x = 1$, $\int_{\mathbf{R}^d} |\nabla G_\sigma(x)|\mathrm{d}x \le C_\sigma$ and $G_\sigma(x) \to \delta_x$ as $\sigma \to 0$, and δ_x is the Dirac measure at point x. Assume that $\tilde{u}$ is an extension of u to $\mathbf{R}^d$ given by periodic reflection through the boundary of Ω. $f : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ is a smooth nondecreasing function and especially in image processing, $f(s) = 0$ if $s \le e$ and $f(s) = 1$ if $s \ge 2e$. The parameter e is the upper bound of the interval where u is allowed to diffuse freely.

The equation is accompanied with zero Dirichlet (or Neumann) boundary and initial conditions as follows:

$$u(\partial u/\partial \nu) = 0 \quad \text{on } I \times \partial\Omega, \tag{1.2}$$

$$u(x,0) = u^0(x) \quad \text{in } \Omega, \tag{1.3}$$

where ν is the unit normal to the boundary of Ω. We assume that the initial function

$$u^0 \in L_\infty(\Omega). \tag{1.4}$$

In image processing (smoothing or de-noising), the initial function $u^0(x)$ represents the grey level intensity of the processed image and the solution $u(t,x)$ is the scaled (filtered, smoothed) version of $u^0(x)$ according to time-scale t.

In the special case of g, $f \equiv 1$, (1.1) reduces to the well-known mean curvature flow level set equation, which has attracted a lot of attention with wide applicability to the geometrical problems such as curve and surface evolution [2–4]. Concerning image processing, the mean curvature equation satisfies the so-called morphological principle on invariance of image analysis to contrast changes and then the process of nonlinear filtration is understood as image multi-scale analysis [5].

The level set like equations have been successively introduced to image processing (image restoration and segmentation) in the last years [1–2,5]. In image filtration, it is generally desirable to smooth the homogeneous regions of the image with the purpose of noise elimination and image interpretation and on the other side, to keep the accurate location of boundaries, i.e., edges. Therefore, it seems natural to modify the diffusion operator so that it diffuses more in direction parallel to the edge and less in the perpendicular one. From such consideration in [1], the authors proposed the image selective smoothing model with preserving edge positions:

$$u_t - g(|\nabla u_\sigma|)|\nabla u|\mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) = 0. \tag{1.5}$$

This model is further improved (see (1.1)) from the fact that it is not necessary to diffuse anisotropically at the points, where gradient is low and moreover to preserve the edges without contrast.