COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
**31**(4)(2015), 362–372

# On Weakly P.P. Rings

XIANG YUE-MING[1], OUYANG LUN-QUN[2] AND WANG SHU-GUI[1]

(*1. Department of Mathematics and Applied Mathematics, Huaihua University, Huaihua, Hunan, 418000*)

(*2. Department of Mathematics, Hunan University of Science and Technology, Xiangtan, Hunan, 411201*)

Communicated by Du Xian-kun

**Abstract:** We introduce, in this paper, the right weakly p.p. rings as the generalization of right p.p. rings. It is shown that many properties of the right p.p. rings can be extended onto the right weakly p.p. rings. Relative examples are constructed. As applications, we also characterize the regular rings and the semisimple rings in terms of the right weakly p.p. rings.

**Key words:** weakly p.p. ring, $GP$-injective module, regular ring

**2010 MR subject classification:** 16D40, 16E50

**Document code:** A

**Article ID:** 1674-5647(2015)04-0362-11

**DOI:** 10.13447/j.1674-5647.2015.04.08

## 1 Introduction

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary. $J(R)$ and $Z(R_R)$ denote, respectively, the Jacobson radical and the right singular ideal of $R$. If $X$ is a subset of $R$, the right (resp. left) annihilator of $X$ in $R$ is denoted by $r(X)$ (resp. $l(X)$). If $X = \{a\}$, we usually abbreviate it to $r(a)$ (resp. $l(a)$). For the usual notations, we refer the readers to [1–3].

A ring $R$ is called Baer (see [2]) if the right annihilator of every nonempty subset of $R$ is generated by an idempotent. The class of Baer rings play a special role in the theory of rings of operators in functional analysis. The notion of p.p. rings is closely related to that of Baer rings. Recall that a ring $R$ is said to be right p.p. (see [4]) (or right Rickart) provided that every principal right ideal is projective, or equivalently the right annihilator of any element of $R$ is a summand of $R_R$. A ring is called a p.p. ring if it is both left and right p.p. ring. We say that an element $a$ of $R$ is right p.p. if $aR$ is projective, or equivalently, if $r(a) = eR$

**Received date:** Dec. 24, 2014.
**Foundation item:** The Scientific Research Foundation (12B101) of Hunan Provincial Education Department.
**E-mail address:** xymls999@126.com (Xiang Y M).

for some $e^2 = e \in R$. Obviously, $R$ is a right p.p. ring if and only if every element of $R$ is right p.p. The p.p. rings and their generalizations have been extensively studied by many authors (see [4–12]). A ring $R$ is called a generalized right p.p. ring if for any $a \in R$, the right ideal $a^nR$ is projective for some positive integer $n$, depending on $a$, or equivalently, if for any $a \in R$, the right annihilator of $a^n$ is generated by an idempotent for some positive integer $n$, depending on $a$.

A right $R$-module $M$ is called $GP$-injective (see [13]) (or $YJ$-injective in [14]) if, for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$ and any right $R$-homomorphism from $a^nR$ to $M$ extends to one from $R$ to $M$. A right $R$-module $N$ is said to be $P$-injective provided that for any $a \in R$, any right $R$-homomorphism from $aR$ to $N$ can extend to one from $R$ to $N$. A ring $R$ is right $GP$-injective (resp. $P$-injective) if $R$ is $GP$-injective (resp. $P$-injective) as a right $R$-module. It was shown that left $GP$-injective rings are the proper generalization of left $P$-injective rings (see [15]). In the recent paper, Mao *et al.*[8] proved that $R$ is a right p.p. ring if and only if every quotient module of any ($P$-)injective right $R$-module is $P$-injective. It was also shown that a ring $R$ is regular if and only if $R$ is a right p.p. and right $P$-injective ring if and only if $R$ is a right p.p. and right $C2$ ring.

This inspires us to develop right weakly p.p. rings. We say that, in this paper, a nonzero element $a$ of $R$ is called right weakly p.p. if there exists a positive integer $n$ such that $a^n \neq 0$ and $a^nR$ is projective, or equivalently, $r(a^n) = eR$ for some $e^2 = e \in R$. The ring $R$ is said to be the right weakly p.p. provided that every nonzero element of $R$ is right weakly p.p. Some examples are given to show that the right weakly p.p. elements need not be the right p.p. elements and the generalized right p.p. rings need not be the right weakly p.p. rings. Many properties of the right p.p. rings are extended onto the right weakly p.p. rings. In Section 3 of the present paper, we investigate the extensions of the right weakly p.p. rings. It is proven that a ring $R$ is right semihereditary if and only if the matrix ring $M_n(R)$ is right weakly p.p. for every $n \geq 1$. Section 4 is devoted to the applications of the right weakly p.p. rings. We characterize (von Neumann) regular rings and the semisimple Artinian rings in terms of the right weakly p.p. rings. Several well-known results are also extended.

## 2 Right Weakly P.P. Rings

We start this section with the definition.

**Definition 2.1** *A nonzero element $a$ of $R$ is called right weakly p.p. if there exists a positive integer $n$ such that $a^n \neq 0$ and $a^nR$ is projective, or equivalently, $r(a^n) = eR$ for some $e^2 = e \in R$. The ring $R$ is said to be right weakly p.p. provided that any nonzero element of $R$ is right weakly p.p. Similarly, we have the concepts of left weakly p.p. elements and rings.*

**Remark 2.1** (1) Obviously, the right p.p. rings are right weakly p.p. and the right weakly p.p. rings are the right generalized p.p. rings.