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The Representive of Metric Projection on the Finite Codimension Subspace in Banach Space

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Abstract: In the paper we introduce the notions of the separation factor κ and give a representive of metric projection on an n-codimension subspace (or an affine set) under certain conditions in Banach space. Further, we obtain the distance formula from any point x to a finite n-codimension subspace. Results extend and improve the corresponding results in Hilbert space.

Key words: n-codimension, separation factor κ, weakly completely separated

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1 Introduction

Let X be a linear normed space and L be a subspace of X. L is called to be n-codimension space if there exists an n-dimension subspace N such that $L \bigoplus N = X$. In Hilbert space, it is well known that there is a formula of the distance from any point x to L, which can be used to solve some optimal control problems (see [1]). In Banach space, Oshman[2] and Fang[3] studied the continuity of metric projection, Fedorov[4] discussed the properties and characterization of the optimal approximation on the subspace of finite codimension in $C[\Omega]$. The representive of the optimal approximation element on a subspace of n-codimension in Banach space is still an unsolved problem. Undoubtedly, one of the main difficulties in dealing with such a problem is that norms in general Banach space are lack of "nice"

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properties such as orthogonality (see [5–6]). Wang *et al.* obtained the representive of the optimal approximation element and the distance formulas from a point x to a hyperplane L (i.e., $n = 1$) (see [7–9]). However, it is more difficult if the dimension of N is larger than 1, a great number of problems on the minimum norm with constrained conditions in optimization and cybernetics can be classified as the case on $n > 1$, for example, calculating $\min\{\|x\|\}$ under constrained conditions $\{\langle y_i^*, x\rangle = c_i,\ y_i^* \in X^*,\ x \in X,\ i = 1, 2, \cdots, n\}$. So it is very meaningful to get a representive of metric projection on a finite n-codimension subspace (or the affine set).

2 Preliminaries

Definition 2.1 *Let X be a Banach space and $L \subset X$. Set*

$$P_L(x) = \{l \in L \mid \|l - x\| = d(x,\ L) = \inf_{l\in L}\{\|l - x\|\},$$

where $d(x,\ L)$ denotes the distance from the point x to L.

Definition 2.2 *Let X be a Banach space and X^* be the dual space of X. The set-valued map $F_X : X \mapsto X^*$ is defined by*

$$F_X(x) = \{x^* \in X \mid \langle x^*,\ x\rangle = \|x^*\|^2 = \|x\|^2\},$$

and the dual map $F_X^{-1} : X^* \mapsto X$ is defined by

$$F_X^{-1}(x^*) = \{x \in X \mid \langle x^*,\ x\rangle = \|x^*\|^2 = \|x\|^2\}.$$

Definition 2.3 *Let X be a reflexive Banach space and X^{**} be the quadratic dual space of X. The typical map $J : X \mapsto X^{**}$ is defined by*

$$J(x) = \{x^{**} \in X^{**} \mid \forall x^* \in X^*,\ \langle x^*,\ x\rangle = \langle x^*,\ x^{**}\rangle\},$$

and

$$J^{-1}(x^{**}) = \{x \in X \mid \forall x^* \in X^*,\ \langle x^*,\ x\rangle = \langle x^*,\ x^{**}\rangle\}.$$

Definition 2.4 *Let X be a Banach space and N be an n-dimension subspace of X. The subspace L is called to be a finite n-codimension if $L \bigoplus N = X$.*

Lemma 2.1 *Let X be a Banach space. Then L is a finite n-codimension subspace if and only if*

$$L = \{x \in X \mid \langle m^*,\ x\rangle = 0,\ \forall m^* \in M^* \subseteq X^*\},$$

where M^* is an n-subspace of X^*.

Definition 2.5 *Assume that E is an n-dimension subspace of the Banach X and $e^* \in E^*$. The extension by which e^* is extended to $\widehat{e^*} \in X^*$ satisfying*

$$\langle \widehat{e^*},\ e\rangle = \langle e^*,\ e\rangle, \qquad e \in E,$$

is called to be a value-preserving prolongation in E. If the value-preserving prolongation of e^* is norm-preserving, specially, we use $\overline{e^*}$ to denote it and say that it is a Hahn-Banach extension.