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Triple Crossing Numbers of Graphs

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Abstract: We introduce the triple crossing number, a variation of the crossing number, of a graph, which is the minimal number of crossing points in all drawings of the graph with only triple crossings. It is defined to be zero for planar graphs, and to be infinite for non-planar graphs which do not admit a drawing with only triple crossings. In this paper, we determine the triple crossing numbers for all complete multipartite graphs which include all complete graphs.

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1 Introduction

Let G be a graph. A drawing of G means a representation of the graph in the Euclidean plane or the 2-sphere, where vertices are points and edges are simple arcs joining their end-vertices. Since each edge is simple, no edge admits self crossings. Furthermore, we assume that the interiors of edges do not contain vertices, and that two edges do not intersect if they have a common vertex, and that two edges without common end-vertex intersect at most once, and if so, then they intersect transversally. These requirements are essential in this paper. A drawing is called a regular drawing (resp. semi-regular drawing) if it has only double (resp. triple) crossing points. From the requirements, we know that a graph has at least 6 vertices if it admits a semi-regular drawing with at least one triple crossing point.

The crossing number $\mathrm{cr}(G)$ of G is defined to be the minimal number of crossing points over all regular drawings of G. In particular, $\mathrm{cr}(G) = 0$ if G is planar. In this paper, we

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introduce a new variation of the crossing number. The triple crossing number $\mathrm{tcr}(G)$ is zero if G is planar, and ∞ if G does not admit a semi-regular drawing. Otherwise, $\mathrm{tcr}(G)$ is defined to be the minimal number of triple crossing points over all semi-regular drawings of G. In particular, $\mathrm{tcr}(G) = 0$ if and only if G is planar.

The triple crossing number can be regarded as a specialization of the degenerate crossing number introduced by Pach and Tóth[1]. In addition, for example, the Petersen graph is known to have the crossing number two (and thus non-planar), and hence has the triple crossing number one from Fig. 1.1. In general, we have the inequality $\mathrm{cr}(G) \leq 3\mathrm{tcr}(G)$ for these two notions, since we obtain a regular drawing from a semi-regular drawing by perturbing each triple crossing point into three double crossing points.

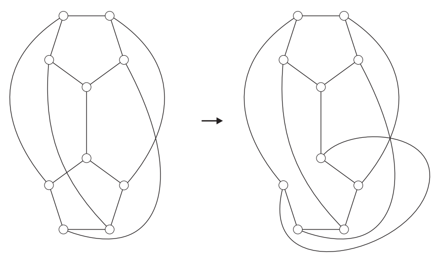

Fig. 1.1 The Petersen graph

In this paper, we determine the triple crossing numbers for all complete multipartite graphs. A complete multipartite graph is a graph whose vertex set can be partitioned into at least two, mutually disjoint non-empty sets, called the partite sets, so that two vertices u and v are adjacent if and only if u and v belong to different sets of the partition. If the partite sets are of sizes $n_1, \cdots, n_t$ $(n_i \geq 1)$, then the graph is denoted by $K_{n_1,\cdots,n_t}$. We always assume that $n_i \geq n_j$ if $i < j$. In particular, if $n_i = 1$ for each i, then the graph $K_{1,\cdots,1}$ is the complete graph K_t with t vertices.

Here is how this paper is organized. After we describe basic lemmas, used in the paper repeatedly, in Section 2, we show that the triple crossing number of a complete t-partite graph is ∞ if $t \geq 5$ in Section 3. In the successive sections, we work on the cases when $t \leq 4$. Here we mention that the hardest part is the case where $t = 2$, in particular, long, but elementary, geometric arguments are needed to show that $K_{5,4}$, $K_{4,4}$, $K_{5,3}$ and $K_{n,3}$ with $n \geq 7$ do not admit a semi-regular drawing. This is treated in Sections 4, 5 and 6. After concluding the case where $t = 2$ in Section 7, the cases where $t = 4$ and $t = 3$ are established in Sections 8 and 9, respectively. Section 10 contains some remarks on our requirements for drawings and a generalization of triple crossing number.

2 Basic Lemmas

Basic terms of graph theory can be found in textbooks such as [2]–[3].

Lemma 2.1 *The complete bipartite graph $K_{3,3}$ and the complete graph K_5 with five vertices are non-planar. Also, a graph is non-planar if it contains $K_{3,3}$ or K_5 as a subgraph.*