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Ore Extensions over Weakly 2-primal Rings

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Abstract: A weakly 2-primal ring is a common generalization of a semicommutative ring, a 2-primal ring and a locally 2-primal ring. In this paper, we investigate Ore extensions over weakly 2-primal rings. Let α be an endomorphism and δ an α-derivation of a ring R. We prove that (1) If R is an (α, δ)-compatible and weakly 2-primal ring, then $R[x;\ \alpha, \delta]$ is weakly semicommutative; (2) If R is (α, δ)-compatible, then R is weakly 2-primal if and only if $R[x;\ \alpha, \delta]$ is weakly 2-primal.

Key words: (α, δ)-compatible ring, weakly 2-primal ring, weakly semicommutative ring, nil-semicommutative ring, Ore extension

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1 Introduction

Throughout this paper, R denotes an associative ring with identity, α is an endomorphism of R and δ is an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for $a, b \in R$. We denote by $R[x;\ \alpha, \delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual, and the multiplication subject to the reaction $xr = \alpha(r)x + \delta(r)$ for any $r \in R$. Particularly, if $\delta = 0_R$, we denote by $R[x; \alpha]$ the skew polynomial ring; if $\alpha = 1_R$, we denote by $R[x; \delta]$ the differential polynomial ring. For a ring R, we denote by $\mathrm{nil}(R)$ the set of all nilpotent elements of R, $\mathrm{Nil}_*(R)$ its lower nil-radical, $\mathrm{Nil}^*(R)$ its upper nil-radical and L-rad(R) its Levitzki radical. For a nonempty subset M of a ring R, the symbol $\langle M\rangle$ denotes the subring (may not with 1) generated by M.

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Recall that a ring R is called reduced if it has no nonzero nilpotent elements; R is symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$; R is semicommutative if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. In [1], semicommutative property is called the insertion-of-factors-property, or IFP. There are many papers to study semicommutative rings and their generalization (see [2]–[5]). Liu and Zhao ([6], Lemma 3.1) has proved that if R is a semicommutative ring, then $\mathrm{nil}(R)$ is an ideal of R. Liang *et al.*[5] called a ring R to be weakly semicommutative if $ab = 0$ implies $aRb \subseteq \mathrm{nil}(R)$ for any $a, b \in R$. This notion is a proper generalization of semicommutative rings by Example 2.2 in [5]. According to Chen[2], a ring R is called nil-semicommutative if $ab \in \mathrm{nil}(R)$ implies $aRb \subseteq \mathrm{nil}(R)$ for any $a, b \in R$. A nil-semicommutative ring is weakly semicommutative, but the converse is not true by Example 2.2 in [2]. Recall that a ring R is 2-primal if $\mathrm{nil}(R) = \mathrm{Nil}_*(R)$. Hong *et al.*[7] called a ring R to be locally 2-primal if each finite subset generates a 2-primal ring, and have shown that if R is a nil ring then R is locally 2-primal if and only if R is a Levitzki radical ring. Chen and Cui[3] called a ring R to be weakly 2-primal if the set of nilpotent elements in R coincides with its Levitzki radical, that is, $\mathrm{nil}(R)$=L-rad(R). Due to Marks[8], a ring R is called NI if $\mathrm{nil}(R) = \mathrm{Nil}^*(R)$. It is obvious that a ring R is NI if and only if $\mathrm{nil}(R)$ forms an ideal, if and only if $R/\mathrm{Nil}^*(R)$ is reduced. Hwang *et al.*[9] considered basic structure and some extensions of NI rings, and Proposition 2.1 in [3] has presented their some characterizations. The following implications hold:

$$\text{Reduced} \Rightarrow \text{Symmetric} \Rightarrow \text{Semicommutative} \Rightarrow \text{2-primal} \Rightarrow \text{Locally 2-primal}$$
$$\Rightarrow \text{Weakly 2-primal} \Rightarrow NI\text{-ring} \Rightarrow \text{Weakly semicommutative.}$$

In general, each of these implications is irreversible (see [3], [7]).

According to Annin[10], for an endomorphism α and an α-derivation δ, a ring R is said to be α-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is called to be δ-compatible if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both α-compatible and δ-compatible, R is called (α, δ)-compatible. Liang *et al.*[5] have proved that if R is α-compatible semicommutative, then $R[x;\ \alpha]$ is weakly semicommutative. Chen and Cui[3] have shown that if R is weakly 2-primal and α-compatible, then $R[x;\ \alpha]$ is weakly 2-primal and hence weakly semicommutative. In this paper, we extend respectively the above results to more general cases, the Ore extensions over weakly 2-primal rings, and generalize recent some related work on polynomial rings and skew polynomial rings. In particular, we show that if R is an (α, δ)-compatible and weakly 2-primal ring, then $R[x;\ \alpha, \delta]$ is a weakly semicommutative ring; if R is (α, δ)-compatible, then R is weakly 2-primal if and only if $R[x;\ \alpha, \delta]$ is weakly 2-primal. At the same time, we also extend a main result proved by Chen[2] to the Ore extensions $R[x;\ \alpha, \delta]$ over weakly 2-primal ring, and obtain that if R is an (α, δ)-compatible and weakly 2-primal ring, then $R[x;\ \alpha, \delta]$ is a nil-semicommutative ring.

In the following, for integers i, j with $0 \le i \le j$, $f_i^j \in \mathrm{End}(R, +)$ denotes the map which is the sum of all possible words in α, δ built with i letters α and $j - i$ letters δ. For instance, $f_2^4 = \alpha^2\delta^2 + \delta^2\alpha^2 + \delta\alpha^2\delta + \alpha\delta^2\alpha + \alpha\delta\alpha\delta + \delta\alpha\delta\alpha$. In particular, $f_0^0 = 1$, $f_i^i = \alpha^i$, $f_0^i = \delta^i$, $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$. For every $f_i^j \in \mathrm{End}(R, +)$ with $0 \le i \le j$, it has C_j^i