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An Identity with Skew Derivations on Lie Ideals

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Abstract: Let R be a 2-torsion free prime ring and L a noncommutative Lie ideal of R. Suppose that (d, σ) is a skew derivation of R such that $x^s d(x) x^t = 0$ for all $x \in L$, where s, t are fixed non-negative integers. Then $d = 0$.

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1 Introduction

Throughout this paper, unless specifically stated, R always denotes a prime ring with center $Z(R)$, Q its Martindale quotient ring. Note that Q is also a prime ring and the center C of Q, which is called the extended centroid of R, is a field (we refer the readers to [1] for the definitions and related properties of these notions). For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. For subsets A, B of R, $[A, B]$ is the additive subgroup generated by all $[a, b]$ with $a \in A$ and $b \in B$. An additive subgroup L of R is said to be a Lie ideal of R if $[l, r] \in L$ for all $l \in L$ and $r \in R$. A Lie ideal L is called noncommutative if $[L, L] \neq 0$. Let L be a noncommutative Lie ideal of R. It is well known that $[R[L, L]R, R] \subseteq L$ (see the proof of Lemma 1.3 in [2]). Since $[L, L] \neq 0$, we have $0 \neq [I, R] \subseteq L$ for $I = R[L, L]R$ a nonzero ideal of R. Recall that a ring R is called prime if for any $x, y \in R$, $xRy = 0$ implies that either $x = 0$ or $y = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Given any

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automorphism σ of R, an additive mapping $d : R \to R$ satisfying

$$d(xy) = d(x)y + \sigma(x)d(y), \qquad x, y \in R$$

is called a σ-derivation of R, or a skew derivation of R with respect to σ, denoted by $(d,\ \sigma)$. It is easy to see that if $\sigma = 1_R$, the identity map of R, then a σ-derivation is merely an ordinary derivation. And if $\sigma \neq 1_R$, then $\sigma - 1_R$ is a skew derivation. Thus the concept of skew derivations can be regarded as a generalization of derivations. When $d(x) = \sigma(x)b - bx$ for some $b \in Q$, then $(d,\ \sigma)$ is called an inner skew derivation, and otherwise it is outer. Any skew derivation $(d,\ \sigma)$ extends uniquely to a skew derivation of Q (see [3]) via extensions of both maps to Q. Thus we may assume that any skew derivation of R is the restriction of a skew derivation of Q. Recall that σ is called an inner automorphism if when acting on Q, $\sigma(q) = uqu^{-1}$ for some invertible $u \in Q$. When σ is not inner, then it is called an outer automorphism. The skew derivations have been extensively studied by many researchers from various views (see for instance [4]–[7] where further references can be found).

A well-known paper of Herstein[2] states that if I is a right ideal of R such that $x^n = 0$ for all $x \in I$, then $I = 0$. Chang and Lin[8] studied a more general case when $d(x)x^n = 0$ and $x^n d(x) = 0$ for all $x \in I$, where d is a nonzero derivation and I is a nonzero right ideal of a prime ring R. Dhara and De Filippis[9] proved the following: Let R be a prime ring, F a generalized derivation of R and L a noncommutative Lie ideal of R. Suppose that $x^s F(x) x^t = 0$ for all $x \in L$, where $s \geq 0$, $t \geq 0$ are fixed integers, then $F = 0$ except when $\mathrm{char} R = 2$ and R satisfies s_4.

In this paper, we continue to investigation on Lie ideals of prime rings, involving a skew derivation $(d,\ \sigma)$ with a nontrivial associated automorphism σ. Here we examine what happens replacing the generalized derivation F by a skew derivation $(d,\ \sigma)$ in the result of [9].

2 Main Results

Theorem 2.1 *Let R be a 2-torsion free prime ring and L be a noncommutative Lie ideal of R. Suppose that $(d,\ \sigma)$ is a skew derivation of R such that $x^s d(x) x^t = 0$ for all $x \in L$, where s, t are fixed non-negative integers. Then $d = 0$.*

Proof. Suppose that $d \neq 0$. We divide the proof into two cases.

Case 1. Suppose that $(d,\ \sigma)$ is X-outer. Set $I = R[L,\ L]R$. Then $0 \neq [I,\ R] \subseteq L$. By the assumption, we have $[x,\ y]^s (d([x,\ y]))[x,\ y]^t = 0$ for all $x, y \in I$ and also for all $x, y \in Q$ by Theorem 2 in [10]. By Theorem 1 in [11], we get

$$[x,\ y]^s (zy + \sigma(x)w - wx - \sigma(y)z)[x,\ y]^t = 0, \qquad x, y, z, w \in Q. \tag{2.1}$$

Subcase 1.1. If σ is X-inner, that is, $\sigma(x) = gxg^{-1}$ for some $g \in Q - C$ since σ is nontrivial. This implies that

$$[x,\ y]^s (zy + gxg^{-1}w - wx - gyg^{-1}z)[x,\ y]^t = 0, \qquad x, y, z, w \in Q. \tag{2.2}$$

Letting $z = 0$ and replacing w by gw in (2.2), we find that

$$[x,\ y]^s g[x,\ w][x,\ y]^t = 0,$$