Communications
in
Mathematical
Research
32(2)(2016), 105–110

Several Hermite-Hadamard Type Inequalities for Harmonically Convex Functions in the Second Sense with Applications

Wang Wen[1,2], Yang Shi-guo[1] and Liu Xue-ying[1]

(*1. School of Mathematics and Statistics, Hefei Normal University, Hefei, 230601*)

(*2. School of Mathematical Science, University of Science and Technology of China, Hefei, 230026*)

Communicated by Wang De-hui

Abstract: In this paper, we first introduce the concept "harmonically convex functions" in the second sense and establish several Hermite-Hadamard type inequalities for harmonically convex functions in the second sense. Finally, some applications to special mean are shown.

Key words: Hermite-Hadamard's inequality, harmonically convex function, mean, inequality

2010 MR subject classification: 26D15, 26A51

Document code: A

Article ID: 1674-5647(2016)02-0105-06

DOI: 10.13447/j.1674-5647.2016.02.02

1 Introduction

Throughout this paper, we let $\mathbf{R} = (-\infty, +\infty)$, $\mathbf{R}_{++} = (0, +\infty)$. We first recall some definitions of various convex functions.

Definition 1.1[1]–[2] *A function $f : I \subset \mathbf{R} \to \mathbf{R}$ is said to be a convex function on I if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \qquad x, y \in I, \ t \in [0, 1];$$

f is a concave function if $-f$ is a convex function.

Definition 1.2[3]–[4] *A function $f : I \subset \mathbf{R}\backslash\{0\} \to \mathbf{R}$ is said to be a harmonically convex function on I if*

$$f\Big(\frac{1}{tx^{-1} + (1-t)y^{-1}}\Big) \le tf(x) + (1-t)f(y), \qquad x, y \in I, \ t \in [0, 1];$$

Received date: Dec. 10, 2014.

Foundation item: The Doctoral Programs Foundation (20113401110009) of Education Ministry of China, Natural Science Research Project (2012kj11) of Hefei Normal University, Universities Natural Science Foundation (KJ2013A220) of Anhui Province, and Research Project of Graduates Innovation Fund (2014yjs02).

E-mail address: wenwang1985@163.com (Wang W).

f is said to be a harmonically concave function if $-f$ is a harmonically convex function.

Definition 1.3[5] *A function $f : I \subset \mathbf{R}_{++} \to \mathbf{R}_{++}$ is said to be an m-AH convex function on I if*

$$f(tx + m(1-t)y) \le \frac{1}{t[f(x)]^{-1} + m(1-t)[f(y)]^{-1}}, \qquad x, y \in I,\ t \in [0,1];$$

f is said to be an m-AH concave function if $-f$ is an m-AH convex function.

Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be a convex function. The following inequality is the well-known Hadamard's inequality

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}, \qquad a, b \in I \text{ with } a < b.$$

We now recall some integral inequalities of Hermite-Hadamard type for some special functions.

Theorem 1.1[3]–[4] *Let $f : I \subset \mathbf{R}\backslash\{0\} \to \mathbf{R}$ be a harmonically convex function, and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then*

$$f\Big(\frac{2ab}{a+b}\Big) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \le \frac{f(a)+f(b)}{2}.$$

For many recent results related to Hermite-Hadamard type inequalities, see [6]–[22].

The aim of this paper is first to introduce the concept "harmonically convex function" in the second sense and establish some Hermite-Hadamard type inequalities for harmonically convex functions in the second sense. Finally, some applications to special mean are shown.

2 Definition and Lemma

The concept of harmonically convex function in the second sense can be introduced as follows.

Definition 2.1[20] *A function $f : I \subset \mathbf{R}\backslash\{0\} \to \mathbf{R}\backslash\{0\}$ is said to be a harmonically convex function in the second sense on I if*

$$f\Big(\frac{1}{tx^{-1} + (1-t)y^{-1}}\Big) \le \frac{1}{t[f(x)]^{-1} + (1-t)[f(y)]^{-1}}, \qquad x, y \in I,\ t \in [0,1]; \tag{2.1}$$

f is said to be a harmonically concave function in the second sense if $-f$ is a harmonically convex function in the second sense.

Lemma 2.1 *Let $f(x) = x^r$ $(x \in \mathbf{R}_{++})$. If $r \le 0$ or $r \ge 1$, then $f(x) = x^r$ is a harmonically concave function in the second sense; If $0 < r < 1$, then $f(x) = x^r$ is a harmonically convex function in the second sense.*

Proof. According to the properties of the function $f(x) = x^r$ $(x \in \mathbf{R}_{++})$, the following results is valid:

(1) For $r \le 0$ or $r \ge 1$, $f(x) = x^r$ is a convex function;